COUPLED FIXED POINT THEOREMS FOR MAPPINGS IN VARIOUS SPACES

THESIS

submitted in fulfilment of the requirement of the degree of

DOCTOR OF PHILOSOPHY

to

YMCA UNIVERSITY OF SCIENCE & TECHNOLOGY

by

MANISH JAIN

Registration No. YMCAUST/Ph09/2011

Under the Supervision of

DR. NEETU GUPTA
ASSOCIATE PROFESSOR
YMCAUST, Faridabad

DR. SANJAY KUMAR
ASSOCIATE PROFESSOR
DCRUST, Murthal (Sonepat)

Department of Mathematics

Faculty of Humanities and Sciences

YMCA UNIVERSITY OF SCIENCE & TECHNOLOGY

Sector-6, Mathura Road, Faridabad, Haryana, India

SEPTEMBER, 2018

DEDICATED

TO

My “GURUJI”

DECLARATION

I hereby declare that this thesis entitled **COUPLED FIXED POINT THEOREMS FOR MAPPINGS IN VARIOUS SPACES** by **MANISH JAIN**, being submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy in DEPARTMENT OF MATHEMATICS under Faculty of Humanities and Sciences of YMCA University of Science & Technology, Faridabad, during the academic year 2018-19, is a bona fide record of my original work carried out under the guidance and supervision of **DR. NEETU GUPTA, ASSOCIATE PROFESSOR, DEPARTMENT OF MATHEMATICS, YMCAUST, FARIDABAD** and **DR. SANJAY KUMAR, ASSOCIATE PROFESSOR, DEPARTMENT OF MATHEMATICS, DCRUST, MURTHAL (SONEPAT)** and has not been presented elsewhere.

I further declare that the thesis does not contain any part of any work which has been submitted for the award of any degree either in this university or in any other university.

(MANISH JAIN)

Registration No. YMCAUST/Ph09/2011

CERTIFICATE

This is to certify that this Thesis entitled **COUPLED FIXED POINT THEOREMS FOR MAPPINGS IN VARIOUS SPACES** by **MANISH JAIN,** submitted in fulfilment of the requirement for the Degree of Doctor of Philosophy in **DEPARTMENT OF MATHEMATICS**, under Faculty of Humanities and Sciences of YMCA University of Science & Technology, Faridabad, during the academic year 2018-19, is a bona fide record of work carried out under our guidance and supervision.

We further declare that to the best of our knowledge, the thesis does not contain any part of any work which has been submitted for the award of any degree either in this university or in any other university.

DR. NEETU GUPTA
ASSOCIATE PROFESSOR
Department of Mathematics
Faculty of Humanities and Sciences
YMCA University of Science & Technology, Faridabad.

DR. SANJAY KUMAR
ASSOCIATE PROFESSOR
Department of Mathematics
DCR University of Science & Technology, Murthal,
Sonepat.

Dated:

ACKNOWLEDGEMENT

First and foremost, I am grateful to the '**Almighty**' for providing me strength and ability to carry out and complete my work.

I pay my deepest gratitude to my sacred spiritual teacher "**GURUJI**" who always showered his blessings onto me.

I express my sincere gratitude to my supervisors **Dr. Neetu Gupta**, Associate Professor, Department of Mathematics, YMCAUST, Faridabad and **Dr. Sanjay Kumar**, Associate Professor, Department of Mathematics, DCRUST, Murthal (Sonepat) for giving me the opportunity to work in this area. It wouldn't have been possible for me to take my thesis to this level without their innovative ideas and relentless support and encouragement.

I wish to thank the Hon'ble Management Committee, Respected Principal Sir, members of the staff and all my students of **Ahir College, Rewari** who always supported me in every possible way.

I would like to express my heartfelt thanks to all my teachers who directed me for higher education. I also avail this opportunity to extend my thanks to **Dr. Ramesh Kumar Vats**, Associate Professor, NIT, Hamirpur (HP), **Dr. Anita Tomar**, Associate Professor, VSKC Government PG College, Dakpathar (Dehradun) and **Dr. Manoj Antil**, Associate Professor, Starex University, Gurugram for their logical suggestions and advices for enhancing my research.

Next, I bow with my folded hands in the feet of my parents **Sh. Ajit Prasad Jain** (my father), **Smt. Shashi Jain** (my mother) and **Sh. Rahul Jain** (my elder brother) and I am deeply beholden to them for their continuous encouragement and support without which it was not possible for me to carry out this work. I also express my thanks to all my friends and well wishers whose names have not been mentioned here due to limitation of the space.

Finally, I express my gratitude to YMCAUST, Faridabad for providing me adequate facilities for my research.

(Manish Jain)

Registration No.- YMCAUST/Ph09/2011

ABSTRACT

Fixed point theory has always been an area of great interest for researchers. The theory is not only limited to metric spaces but is also enjoyed in various spaces including partial metric spaces, G-metric spaces, Menger PM-Spaces, PGM-Spaces, fuzzy metric spaces, etc. Recently, this theory has been receiving attention of authors in various spaces equipped with a partial order. Now-a-days, an important branch of this theory, popularly known as "coupled fixed point theory" is being readily explored by researchers.

Over last few decades, several interesting results involving distinct contractive/ contraction conditions have been formulated by different authors in various spaces. Present work deals with the investigation of coupled fixed points for mappings subjected to different conditions in various spaces. Our aim is to generalize and extend the already existing works present in the literature. The contractions are designed by us in such a way that weakens some notable works of different authors.

During literature review, we came across some errors and omissions. Correcting errors in the existing work requires counter examples and strong arguments. We have provided proper illustrations to support our arguments while correcting the errors and omissions in the existing work.

Authors are continuously framing their results using different techniques. These techniques require a proper analysis for implementation in one's own work. While framing our results, it has been found that a recently developed technique to compute coupled coincidence points may be improved and the improvement has been provided in the current work.

Different authors have employed distinct conditions on the mappings and the spaces to formulate their fixed point/ common fixed point/ coupled fixed point/ coupled common fixed point results. Among these conditions the condition of continuity, commutativity, compatibility, the containment of range spaces of the involved mappings into one another, the completeness of the space or range subspaces are the main assumptions taken into account by researchers to develop their results.

In the present work, we have tried to relax and replace some of the above mentioned conditions by some more natural conditions.

Very recently, authors have introduced new notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property in fixed point theory. In the present work, we have also designed these notions for problems in coupled fixed point theory.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS AND NOTATIONS

1.	$\mathbb{R}$	Set Of All Real Numbers
2.	poset	Partially Ordered Set
3.	w.r.t.	With Respect To
4.	POMS	Partially Ordered Metric Space
5.	$\mathbb{R}^+$	Set Of All Non-Negative Real Numbers
6.	POPMS	Partially Ordered Partial Metric Space
7.	POGMS	Partially Ordered G-Metric Space
8.	$\mathbb{N}$	Set Of All Natural Numbers
9.	Λ^+	Set Of All Menger Distance Distribution Functions
10.	POMPMS	Partially Ordered Menger PM-Space
11.	KMFMS	KM-Fuzzy Metric Space
12.	GVFMS	GV-Fuzzy Metric Space
13.	BCP	Banach Contraction Principle
14.	$\mathbb{R}^+ \setminus \{0\}$	Set Of All Positive Real Numbers
15.	CF-Ψ	Family Of All Comparison Functions
16.	CCF- Ψ	Family Of All (c)-Comparison Functions
17.	POCMS	Partially Ordered Complete Metric Space
18.	ADF	Altering Distance Function
19.	MMP	Mixed Monotone Property
20.	MgMP	Mixed g-Monotone Property
21.	POCPMS	Partially Ordered Complete Partial Metric Space
22.	POCGMS	Partially Ordered Complete G-Metric Space
23.	POCMPMS	Partially Ordered Complete Menger PM-Space
24.	W.L.O.G.	Without Loss Of Generality
25.	MSMP	Mixed Strict Monotone Property
26.	MSgMP	Mixed Strict g-Monotone Property
27.	WC	Weakly commuting
28.	R-WC	R-weakly commuting
29.	R-WC(A_F)	R-weakly commuting of type (A_F)
30.	R-WC(A_g)	R-weakly commuting of type (A_g)

31.	R-WC(P)	R-weakly commuting of type (P)
32.	COM(A)	Compatible of type (A)
33.	COM(B)	Compatible of type (B)
34.	COM(P)	Compatible of type (P)
35.	COM(C)	Compatible of type (C)
36.	COM(A_F)	Compatible of type (A_F)
37.	COM(A_g)	Compatible of type (A_g)

CHAPTER – I

INTRODUCTION

1.1 AN OVERVIEW OF FIXED POINT THEORY

In mathematics, very often situation arises where the solutions of a system of equations cannot be found in an explicit and convenient way. Naturally, some general questions arise, viz.

"Does the given system of equations has a solution?"

"How many different solutions exist for the given system of equations?"

These questions after being answered are followed by a new question, "If the solution exists, what is the exact or approximate solution of the given system of equations?" This leads to the origination of the theory of fixed points. The problem of solving the system of equations can be reduced to the problem of computing the fixed points or common fixed points of self mapping(s) defined over some appropriate space Ҳ.

Mathematically, the point of intersection of the curve y = ħϰ with the line y = ϰ yields the fixed point of the curve y = ħϰ. In particular, the solution of the equation ħϰ = ϰ gives the fixed point for the self mapping ħ defined on the abstract set Ҳ. A point ɑ ∈ Ҳ is called a fixed point of the mapping ħ defined on Ҳ, if it remains invariant under the mapping ħ, that is, if ħɑ = ɑ.

The theory dealing with fixed points of certain mapping(s) is called fixed point theory and can be seen as a fair combination of topology, analysis and geometry. Over the last five decades, this theory has been used as an important and dominant tool to study the phenomena of nonlinear analysis. The theory has a wide range of applications in various disciplines including physics, chemistry, biology, engineering, economics, game theory etc.

For the last quarter of the twentieth century, there has been a considerable interest among researchers to study fixed points for mappings satisfying certain contractive or contraction conditions in various spaces. Several interesting results concerning the computation of fixed points have been established in various spaces. Now-a-days, authors are not only taking interest in developing fixed point results for self mappings but also for the mappings with domain as the product space Ҳ × Ҳ and co-domain being the space Ҳ, under consideration. The theory of fixed points dealing with such mappings is, however, called coupled fixed point theory and the fixed points for such mappings are called coupled fixed points. Present study deals with the computation of coupled fixed points in various spaces. In the subsequent sections of this chapter, we will study some spaces (metric spaces, partial metric spaces, G-metric spaces, Menger PM-Spaces, PGM-Spaces and fuzzy metric spaces) in which we will develop our results in the subsequent chapters. Collectively, we call these spaces as abstract spaces.

1.2 METRIC SPACES AND PARTIALLY ORDERED SETS

In the study of fixed point theory, the notion of metric plays an important role. The word metric has actually been derived from the word metor, which means measure. In 1906, the famous French mathematician, M. Frechet (1878-1973), in his doctoral thesis submitted to the University of Paris, pioneered the notion of metric spaces.

Definition 1.2.1. Let X be a non-empty set and $d: X \times X \to \mathbb{R}$ be a function such that for $x, y, z \in X$, the following conditions hold:

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0 \Leftrightarrow x = y$;

(iii) $d(x, y) = d(y, x)$; (symmetric property)

(iv) $d(x, y) \leq d(x, z) + d(z, y)$. (triangular inequality)

The function d is called a **metric** on X and together with X is called the **metric space**, represented by (X, d). The elements of X are called points and the function $d(x, y)$ denotes the distance between the points x and y.

For instance, let $X = \mathbb{R}$ (the set of real numbers) and $d: X \times X \to \mathbb{R}$ be a function defined by $d(x, y) = |x - y|$ for $x, y \in X$, then (X, d) is a metric space and this metric is popularly known as the **usual metric**.

In a mathematical system, partially ordered set (**poset**) signifies the idea of ordering of elements of a set. A poset consists of a set together with a binary relation w.r.t. which for certain pairs of elements in the set, one of the element precedes the other (and such elements are called **comparable**). Such a relation is called a **partial order**. In a poset, for some pairs of elements, it may also happen that neither element precedes the other.

Formally, we now state some definitions.

Definition 1.2.2. An **ordered pair** is a pair of objects or elements taken in a specific order. For example, (y, ⱬ) is an ordered pair in y and ⱬ, where y is called the **first element** and ⱬ is called the **second element**.

Definition 1.2.3. Let ϓ and Ⱬ be two non-empty sets. The **cartesian product** of ϓ and Ⱬ, denoted by ϓ × Ⱬ, is the set of all the ordered pairs (y, ⱬ) in which the first element y is from the set ϓ and the second element ⱬ is from the set Ⱬ.

In symbols, we write ϓ × Ⱬ = {(y, ⱬ): y ∈ ϓ and ⱬ ∈ Ⱬ}.

Definition 1.2.4. A **binary relation** on a set ϓ is the collection of ordered pairs of elements of ϓ.

Definition 1.2.5. A **partial order** or **non-strict partial order** is a binary relation $\preccurlyeq$ over a set Ӽ which satisfies for all ϰ, y, ⱬ in Ӽ, the following conditions:

(i) $\varkappa \preccurlyeq \varkappa$;

(ii) if $\varkappa \preccurlyeq y$ and $y \preccurlyeq \varkappa$, then $\varkappa = y$;

(iii) if $\varkappa \preccurlyeq y$ and $y \preccurlyeq z$, then $\varkappa \preccurlyeq z$.

A **poset** is defined as, "A set with a partial order". In general, if Ӽ is a non-empty set with a partial order $\preccurlyeq$, then we denote the **poset** by (Ӽ, $\preccurlyeq$). Further, the elements ϰ, y of a poset (Ӽ, $\preccurlyeq$) are said to be **comparable** if either $\varkappa \preccurlyeq y$ or $y \preccurlyeq \varkappa$.

Definition 1.2.6. A **strict partial order** $\prec$ is a binary relation over a set Ӽ which satisfies for all ϰ, y, ⱬ in Ӽ, the following conditions:

(i) not $\varkappa \prec \varkappa$;

(ii) if $\varkappa \prec y$ and $y \prec z$, then $\varkappa \prec z$;

(iii) if $\varkappa \prec y$, then not $y \prec \varkappa$.

Also, for a partial order $\preccurlyeq$ on the non-empty set Ӽ, the strict partial order $\prec$ on Ӽ is defined as $\varkappa \prec y$, which means that $\varkappa \preccurlyeq y$ but $\varkappa \neq y$ for ϰ, y in Ӽ.

The **inverse** or **converse** $\succcurlyeq$ of a partial order relation $\preccurlyeq$ is said to satisfy $\varkappa \succcurlyeq y$ iff $y \preccurlyeq \varkappa$. Clearly, the inverse of a partial order relation is itself a partial order relation. The order dual of a poset is the same set with the partial order relation replaced by its own inverse.

Definition 1.2.7. A **total order** or **linear order** is a binary relation $\preccurlyeq$ over a set Ӽ which satisfies for all ϰ, y, ⱬ in Ӽ, the following conditions:

(i) if $\varkappa \preccurlyeq y$ and $y \preccurlyeq \varkappa$, then $\varkappa = y$;

(ii) if $\varkappa \preccurlyeq y$ and $y \preccurlyeq z$, then $\varkappa \preccurlyeq z$;

(iii) $\varkappa \preccurlyeq y$ or $y \preccurlyeq \varkappa$.

A set paired with a total order is called a **totally ordered set**.

Definition 1.2.8. Let $\acute{Y}$ be a subset of a poset $(X, \preccurlyeq)$, then

(i) an element $l \in \acute{Y}$ is called a **lower bound** of $\acute{Y}$ iff

$$l \preccurlyeq \varkappa \text{ for all } \varkappa \in \acute{Y};$$

(ii) an element $u \in \acute{Y}$ is called an **upper bound** of $\acute{Y}$ iff

$$\varkappa \preccurlyeq u \text{ for all } \varkappa \in \acute{Y}.$$

Definition 1.2.9. A self mapping $\hbar$ defined on a poset $(X, \preccurlyeq)$ is called

(i) **order preserving** (**monotonically increasing**), if

for $\varkappa, y \in X$ with $y \preccurlyeq \varkappa$, we have $\hbar y \preccurlyeq \hbar\varkappa$;

(ii) **order reversing** (**monotonically decreasing**), if

for $\varkappa, y \in X$ with $y \preccurlyeq \varkappa$, we have $\hbar y \succcurlyeq \hbar\varkappa$;

(iii) **strictly increasing**, if

for $\varkappa, y \in X$ with $y \prec \varkappa$, we have $\hbar y \prec \hbar\varkappa$;

(iv) **strictly decreasing**, if

for $\varkappa, y \in X$ with $y \prec \varkappa$, we have $\hbar y \succ \hbar\varkappa$.

A self mapping $\hbar$ defined on a poset $(X, \preccurlyeq)$ is called **monotone** if it is either order preserving or order reversing.

If $(X, \preccurlyeq)$ is a poset, then the relation $\sqsubseteq$ defined on $X \times X$ by

$$(\varkappa, y) \sqsubseteq (u, v) \Leftrightarrow \varkappa \preccurlyeq u,\ y \succcurlyeq v,$$

for $(\varkappa, y), (u, v) \in X \times X$, is also a partial order relation and $(X \times X, \sqsubseteq)$ is a poset.

If we have $(u, v) \sqsubseteq (\varkappa, y)$, then we may also write $(\varkappa, y) \sqsupseteq (u, v)$. In this case, we say that

$$(\varkappa, y) \sqsupseteq (u, v) \Leftrightarrow \varkappa \succcurlyeq u,\ y \preccurlyeq v.$$

For the sake of convenience, we use the symbol $\preccurlyeq$ in place of $\sqsubseteq$ and $\succcurlyeq$ in place of $\sqsupseteq$. Now, we say that $(\varkappa, y)$ and (u, v) are comparable if $(\varkappa, y) \preccurlyeq (u, v)$ or $(\varkappa, y) \succcurlyeq (u, v)$.

Now-a-days, researchers are giving much attention to the partially ordered metric space **(POMS)**. POMS refers to a metric space endowed with a partial order. If (X, d) is a metric space and "$\preccurlyeq$" is a partial order on X, then POMS is represented by $(X, \preccurlyeq, d)$ and can be defined as:

Let X be a non-empty set. Then, $(X, \preccurlyeq, d)$ is called a **POMS** if:

(i) $(X, \preccurlyeq)$ is a poset; (ii) (X, d) is a metric space.

1.3 PARTIAL METRIC SPACES

In 1994, Matthews [1] introduced the concept of partial metric spaces as a generalization of metric spaces, in which self distance of a point may not be zero. According to Matthews, "Metric spaces are certainly Hausdorff and consequently, cannot be used to study non-Hausdorff topologies". In fact, Matthews [1] introduced an approach to extend metric tools to non-Hausdorff topologies. The notion of partial metric spaces given by Matthews [1] is as follows:

Definition 1.3.1 ([1]). A **partial metric** on a non-empty set Ӽ is a function þ: Ӽ × Ӽ → $\mathbb{R}^+$ such that for all ϰ, y, z in Ӽ, the following holds:

þ1. ϰ = y ⟺ þ(ϰ, ϰ) = þ(ϰ, y) = þ(y, y);

þ2. þ(ϰ, ϰ) ≤ þ(ϰ, y);

þ3. þ(ϰ, y) = þ(y, ϰ);

þ4. þ(ϰ, y) ≤ þ(ϰ, z) + þ(z, y) − þ(z, z).

A **partial metric space** is a pair (Ӽ, þ) such that the set Ӽ is non-empty and þ is a partial metric on Ӽ. Clearly, if þ(ϰ, y) = 0, then ϰ = y. But the distance of any point from itself need not be zero.

Alike to the fixed point results in POMS, authors are also formulating fixed point and coupled fixed point results in the partially ordered partial metric space (**POPMS**). In general, POPMS refers to a partial metric space (Ӽ, þ) endowed with the partial order ≼ and is represented by (Ӽ, ≼, þ). POPMS can be defined as:

Let Ӽ be a non-empty set. Then, (Ӽ, ≼, þ) is called a **POPMS** if:

(i) (Ӽ, ≼) is a poset; (ii) (Ӽ, þ) is a partial metric space.

1.4 G-METRIC SPACES

In order to generalize the notion of distance, Gahler [2] in 1963 introduced the concept of 2-metric spaces. Afterwards, several fixed point results came into existence in these spaces. Hsiao [3] showed that all such results were trivial. Later on, Ha et al. [4] proved that a 2-metric need not be a continuous function in its variables, whereas an ordinary metric is, further there has been no easy relationship between the results obtained in the setting of these two structures.

On the other hand, in 1984, B.C. Dhage [5] in his doctral thesis introduced the concept of D-metric spaces as a generalization of ordinary metric space. Corresponding to every metric space, there exists a D-metric space. The converse is however, not true in general. Geometrically, a 2-metric represents the area of a triangle, whereas a D-metric represents the perimeter of a triangle.

In 2003, Mustafa and Sims [6] demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces were incorrect. In order to overcome the weaknesses of Dhages's theory, Mustafa and Sims [7] in 2006 formulated a more vital generalization of metric spaces, termed as the generalized metric space (**G-metric space**).

Definition 1.4.1 ([7]). Let $\mathfrak{X}$ be a non-empty set and G: $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(\varkappa, y, z) = 0$ if $\varkappa = y = z$;

(G2) $0 < G(\varkappa, \varkappa, y)$ for all $\varkappa, y \in X$ with $\varkappa \neq y$;

(G3) $G(\varkappa, \varkappa, y) \leq G(\varkappa, y, z)$ for all $\varkappa, y, z \in X$ with $z \neq y$;

(G4) $G(\varkappa, y, z) = G(\varkappa, z, y) = G(y, z, \varkappa) = \ldots$ (symmetry in all three variables);

(G5) $G(\varkappa, y, z) \leq G(\varkappa, a, a) + G(a, y, z)$ for all $\varkappa, y, z, a \in \mathfrak{X}$.

(rectangle inequality)

Then, the function G is called a **generalized metric** on $\mathfrak{X}$ (**G-metric** on $\mathfrak{X}$) and the pair $(\mathfrak{X}, G)$ is termed as a **G-metric space**.

Fixed point theory in this structure was initiated by Mustafa et al. [8], following which, different authors proved several fixed point results in this set up.

Following the recent trends in fixed point theory, researchers are also enjoying fixed point and coupled fixed point results in the partially ordered G-metric space (**POGMS**). POGMS refers to the G-metric space $(\mathfrak{X}, G)$ endowed with a partial order $\preccurlyeq$ and is represented by $(\mathfrak{X}, \preccurlyeq, G)$. It can be defined as:

Let $\mathfrak{X}$ be a non-empty set. Then, $(\mathfrak{X}, \preccurlyeq, G)$ is called a **POGMS** if:

(i) $(\mathfrak{X}, \preccurlyeq)$ is a poset; (ii) $(\mathfrak{X}, G)$ is a G-metric space.

1.5 MENGER PM-SPACES AND PGM-SPACES

For years, researchers are continuously making efforts to generalize the structure of metric space under different conditions. There have been a number of generalizations of metric space out of which, an important one is the Menger probabilistic metric space (**Menger PM-space** or **Menger space**). In 1942, the study of probabilistic metric space (**PM-space**) was initiated by Menger [9] under the name of statistical metrics. Since then, the theory of PM-spaces has been developed in many directions, particularly by Schweizer and Sklar [10, 11] as Menger PM-spaces. In fact, the PM-space is the probabilistic generalization of the metric space in which a distribution function $F_{\varkappa, y}$ is associated with every pair of elements $\varkappa$, y rather than associating the distance $d(\varkappa, y)$ between $\varkappa$ and y. For $\mathfrak{t} > 0$, $F_{\varkappa, y}$ represents the probability that the distance between $\varkappa$ and y is less than $\mathfrak{t}$. The perception of PM-space corresponds to those situations where the distance between two points is not known exactly but we know the probabilities of the possible values of the distance. This probabilistic generalization of metric spaces is of fundamental importance in probabilistic functional analysis [12].

In 1966, Sehgal [13] in his doctoral dissertation initiated the study of fixed points in PM-spaces by proving the contraction principle in these spaces. Afterwards, various authors have done much work in these spaces.

Definition 1.5.1 ([11]). A function $f\colon \mathbb{R}^+ \to [0, 1]$ is called a **distribution function** if it is left-continuous and non-decreasing with $\inf_{\varkappa\in\mathbb{R}} f(\varkappa) = 0$. If addionally, $f(0) = 0$, then f is called a **distance distribution function**. A distance distribution function f that satisfies $\lim_{\mathfrak{t}\to\infty} f(\mathfrak{t}) = 1$ is called a **Menger distance distribution function**. The set of all Menger distance distribution functions is denoted by Λ^+.

Definition 1.5.2 ([10, 11]). A **triangular norm** (**t-norm**) is a binary operation $\Delta\colon [0, 1] \times [0, 1] \to [0, 1]$ satisfying:

(i) $\Delta(a, b) = \Delta(b, a)$;

(ii) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$;

(iii) $\Delta(a, 1) = a$;

(iv) $\Delta(a, b) \leq \Delta(c, d)$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

A t-norm is **continuous** if it is continuous as a function. A t-norm Δ is said to be **positive** if $\Delta(a, b) > 0$ for all $a, b \in (0, 1]$. Some examples of the continuous t-norm are $\Delta_p(a, b) = ab$ and $\Delta_m(a, b) = \min\{a, b\}$ for $a, b \in [0, 1]$.

Note that, a t-norm can also be denoted by the symbol $*$.

Definition 1.5.3 ([14]). Let $\sup_{0<\mathfrak{t}<1} \Delta(\mathfrak{t}, \mathfrak{t}) = 1$. A t-norm Δ is said to be a **Hadžić type t-norm** (**H-type t-norm** or **t-norm of H-type**), if the family of functions $\{\Delta^p(\mathfrak{t})\}_{p=1}^{\infty}$ is equi-continuous at $\mathfrak{t} = 1$, where $\Delta^{p+1}(\mathfrak{t}) = \Delta(\mathfrak{t}, \Delta^p(\mathfrak{t})) = \mathfrak{t}\, \Delta\, (\Delta^p(\mathfrak{t}))$, p = 1, 2, ... and $\mathfrak{t} \in [0, 1]$.

The t-norm Δ_m is an example of t-norm of H-type.

Remark 1.5.1 ([14]). A t-norm Δ is a H-type t-norm iff for any $\sigma \in (0, 1)$, there exists $\varrho(\sigma) \in (0, 1)$ such that $\Delta^{p}(\mathfrak{t}) > (1 - \sigma)$ for all $p \in \mathbb{N}$, when $\mathfrak{t} > (1 - \varrho)$.

Definition 1.5.4 ([10, 11]). A **Menger PM-space** is a triple $(\mathfrak{X}, F, \Delta)$, where $\mathfrak{X}$ is a non-empty set, Δ is a continuous t-norm and F is a mapping from $\mathfrak{X} \times \mathfrak{X}$ into Λ^{+} such that, if $F_{\varkappa,y}$ denotes the value of F at the pair $(\varkappa, y)$, the following conditions hold for all $\varkappa, y, z \in \mathfrak{X}$ and $\mathfrak{t}, s > 0$:

(PM$_1$) $F_{\varkappa,y}(\mathfrak{t}) = 1$ iff $\varkappa = y$;

(PM$_2$) $F_{\varkappa,y}(\mathfrak{t}) = F_{y,\varkappa}(\mathfrak{t})$;

(PM$_3$) $F_{\varkappa,z}(\mathfrak{t} + s) \geq \Delta\big(F_{\varkappa,y}(\mathfrak{t}), F_{y,z}(s)\big)$.

In present work, a partially ordered Menger PM-space (**POMPMS**) refers to the Menger PM-space $(\mathfrak{X}, F, \Delta)$ endowed with a partial order $\preccurlyeq$ and is represented by $(\mathfrak{X}, \preccurlyeq, F, \Delta)$. POMPMS can be defined as:

Let $\mathfrak{X}$ be a non-empty set. Then, $(\mathfrak{X}, \preccurlyeq, F, \Delta)$ is called a **POMPMS** if:

(i) $(\mathfrak{X}, \preccurlyeq)$ is a poset; (ii) $(\mathfrak{X}, F, \Delta)$ is a Menger PM-space.

Recently, Zhou et al. [15] provided a probabilistic version of G-metric spaces and proved some fixed point results in it.

Definition 1.5.5 [15]. A **Menger probabilistic G-metric space** (**PGM-space**) is a triple $(\mathfrak{X}, G^{*}, \Delta)$, where $\mathfrak{X}$ is a non-empty set, Δ is a continuous t-norm and G^{*} is a mapping from $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ into Λ^{+} ($G^{*}_{\varkappa,y,z}$ denote the value of G^{*} at the point $(\varkappa, y, z)$) satisfying the following conditions for all $\varkappa, y, z, \mathfrak{a} \in \mathfrak{X}$ and $\mathfrak{t}, s > 0$:

(PGM-1) $G^{*}_{\varkappa,y,z}(\mathfrak{t}) = 1$ iff $\varkappa = y = z$;

(PGM-2) $G^{*}_{\varkappa,\varkappa,y}(\mathfrak{t}) \geq G^{*}_{\varkappa,y,z}(\mathfrak{t})$ where $y \neq z$;

(PGM-3) $G^{*}_{\varkappa,y,z}(\mathfrak{t}) = G^{*}_{\varkappa,z,y}(\mathfrak{t}) = G^{*}_{y,\varkappa,z}(\mathfrak{t}) = \ldots$;

(PGM-4) $G^{*}_{\varkappa,y,z}(\mathfrak{t} + s) \geq \Delta\left(G^{*}_{\varkappa,\mathfrak{a},\mathfrak{a}}(s), G^{*}_{\mathfrak{a},y,z}(\mathfrak{t})\right)$.

As in Menger PM-spaces, the theory of fixed points is growing rapidly in PGM-spaces also.

1.6 FUZZY METRIC SPACES

In 1965, Zadeh [16] lead the beginning of a new era by introducing the concept of fuzzy sets. The abstraction of the notion of distance under fuzzy situation has been stimulated by various authors in distinct ways (see, Deng [17], Erceg [18], Kaleva and

Seikkala [19], Kramosil and Michalek [20], George and Veeramani [21, 22]). In 1975, Kramosil and Michalek [20] introduced the concept of fuzzy metric spaces. Later on, George and Veeramani [21, 22] with a view point to obtain the Hausdorff topology in these spaces modified the concept of fuzzy metric spaces due to Kramosil and Michalek [20].

Afterwards, various authors established several fixed point results in fuzzy metric spaces in the sense of George and Veeramani [21, 22], one can refer for more details the work done by Gregori and Sapena [23], Murthy et al. [24], Singh and Chauhan [25], etc.

Definition 1.6.1 ([16]). A **fuzzy set** A in $\mathcal{X}$ is a function with domain $\mathcal{X}$ and values in [0, 1].

Definition 1.6.2. ([20]). A **fuzzy metric space** in the sense of Kramosil and Michalek (**KM-fuzzy metric space**) is a triple $(\mathcal{X}, \mathcal{M}, *)$, where $\mathcal{X}$ is a non-empty set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathcal{X}^2 \times \mathbb{R}^+$ satisfying for all $\varkappa, y, z \in \mathcal{X}$ and $t, s > 0$, the following conditions:

(KM-1) $\mathcal{M}(\varkappa, y, 0) = 0$;

(KM-2) $\mathcal{M}(\varkappa, y, t) = 1$ iff $\varkappa = y$;

(KM-3) $\mathcal{M}(\varkappa, y, t) = \mathcal{M}(y, \varkappa, t)$;

(KM-4) $\mathcal{M}(\varkappa, z, t + s) \geq \mathcal{M}(\varkappa, y, t) * \mathcal{M}(y, z, s)$;

(KM-5) $\mathcal{M}(\varkappa, y, .): \mathbb{R}^+ \to [0, 1]$ is left continuous.

George and Veeramani [21, 22] modified this notion of fuzzy metric spaces as follows:

Definition 1.6.3 ([21, 22]). The 3-tuple $(\mathcal{X}, \mathcal{M}, *)$ is called a **fuzzy metric space** in the sense of George and Veeramani (**GV-fuzzy metric space**) if $\mathcal{X}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathcal{X}^2 \times \mathbb{R}^+ \setminus \{0\}$ satisfying the following conditions for each $\varkappa, y, z \in \mathcal{X}$ and $t, s > 0$:

(FM-1) $\mathcal{M}(\varkappa, y, t) > 0$;

(FM-2) $\mathcal{M}(\varkappa, y, t) = 1$ iff $\varkappa = y$;

(FM-3) $\mathcal{M}(\varkappa, y, t) = \mathcal{M}(y, \varkappa, t)$;

(FM-4) $\mathcal{M}(\varkappa, z, t + s) \geq \mathcal{M}(\varkappa, y, t) * \mathcal{M}(y, z, s)$;

(FM-5) $\mathcal{M}(\varkappa, y, .): \mathbb{R}^+ \setminus \{0\} \to [0, 1]$ is continuous.

For brevity, we call KM-fuzzy metric space as **KMFMS** and GV-fuzzy metric space as **GVFMS**.

Some authors including Jain et al. [26], Choudhury et al. [27], Choudhury and Das [28], and others, have also used the following additional condition to formulate their results in GVFMS:

(FM-6) $M(\varkappa, y, t) \to 1$ as $t \to \infty$ for $\varkappa, y \in X$.

Presently, authors are formulating fixed point results in these spaces enthusiastically.

1.7 OBJECTIVE OF THE PRESENT WORK

- We aim to extend, unify and generalize the results of various authors present in the literature of coupled fixed point theory in various abstract spaces.
- Try to define nonlinear contractions in such-a-way that extend and generalize the previous results present in the literature.
- Try to improve the technique used to compute coupled coincidence points.
- Try to modify and rectify errors present in the already existing results.
- To study the relation between the ordinary fixed point and coupled fixed point results.
- Recently, Aamri and El-Moutawakil [29] introduced the concept of property (E.A.) and subsequently, Sintunavarat and Kumam [30] introduced the concept of (CLRg) property for obtaining fixed points. We aim to study these notions for the problems in coupled fixed point theory and to extend these notions to common property (E.A.) and (CLR_{ST}) property for problems in coupled fixed point theory.

1.8 METHODOLOGY ADOPTED FOR THE PRESENT WORK

Banach fixed point theorem is a fundamental tool in "fixed point theory", which guarantees the existence and uniqueness of fixed points of certain self map(s) on metric spaces and thereby provides a constructive method to find fixed points. Generally, the following steps are followed:

Step 1. To find a common coincidence point for one pair of maps;

Step 2. To find a common coincidence point for the second pair using 1st step;

Step 3. To show that pair wise coincidence points are equal;

Step 4. To show that common coincidence point is a common fixed point;

Step 5. To show the uniqueness.

Further, the Inductive, Deductive, Heuristic, Analytic and Synthetic approaches are also used to prove the results.

1.9 ORGANIZATION OF PRESENT WORK

Present work deals with the aim to fulfil the objectives mentioned in section 1.7. The work is divided into eight chapters and each chapter has many sub-sections.

CHAPTER – I is the introduction part and consists of nine sections. Section 1.1 provides an overview of the fixed point theory. In section 1.2, we study the notions of metric spaces and partially ordered sets. Section 1.3 accounts the partial metric spaces. In section 1.4, we study the notion of G-metric spaces. Section 1.5 gives an introduction to Menger PM-spaces and PGM-spaces. Similarly, section 1.6 gives the introduction of fuzzy metric spaces. In section 1.7, we discuss the objectives of the current study. Section 1.8 accounts the methodology adopted for the present work. In section 1.9, organization of the present work is given.

CHAPTER – II provides a deep insight into the literature review which motivates to carry out the present research. It consists of five sections. Section 2.1 provides a metrical survey of fixed point theory which comprehends fixed point results as well as coupled fixed point results in POMS. Section 2.2 indulges literature survey in partial metric spaces. The survey of literature in G-metric spaces has been presented in section 2.3. Section 2.4 grants the literature review of coupled fixed point theory in Menger PM-spaces and PGM-spaces. Section 2.5 corresponds the relevant analysis of literature in fuzzy metric spaces.

CHAPTER – III deals with (φ, ψ) – contractive conditions in POMS and POPMS. The contractive conditions under consideration are symmetric in nature and weaken some of the already existing contractive conditions present in the literature. This chapter consists of five sections. Section 3.1 gives a brief introduction to symmetric contractive conditions. In section 3.2, we establish the existence and uniqueness of coupled common fixed points under a (φ, ψ) – contractive condition for mappings with mixed g-monotone property (**MgMP**) in POMS. Section 3.3 consists of coupled fixed point results under a (φ, ψ) – contractive condition in POMS. In section 3.4, we establish coupled fixed point result for symmetric (φ, ψ) – weakly contractive condition in the setup of POPMS. In the last section 3.5, an application to the existence and uniqueness of the solution of an integral equation is discussed. In this section, a result of the integral type is also given.

CHAPTER – IV deals with some generalized and weak symmetric contractions in POMS. This chapter has five sections. Section 4.1 gives a brief introduction to some symmetric contractions. In section 4.2, we establish some coupled common fixed point results under the notion of generalized symmetric g-Meir-Keeler type contractions. Section 4.3 consists of coupled common fixed point results for mixed g-monotone mappings satisfying (α, ψ) – weak contractions. In section 4.4, as applications of the results proved in various sections of this chapter, the solution of integral equations is discussed. In the last section 4.5, an application to the result of the integral type is also given.

In **CHAPTER – V**, we establish some coupled coincidence and coupled common fixed point results in the setup of POGMS for mixed g-monotone mappings. The results obtained generalize and extend works of various authors present in the literature. This chapter consists of four sections. Section 5.1 gives a brief introduction to coupled fixed point results in G-metric spaces. In section 5.2, we establish some coupled coincidence and coupled common fixed point results for mixed g-monotone mappings satisfying $(\phi, \psi) -$ contractive conditions in POGMS. Section 5.3 consists of some coupled coincidence and coupled common fixed point results for mixed g-monotone mappings satisfying new generalized nonlinear contractions in the setup of POGMS. At last, in section 5.4, as application of the obtained results, we discuss the solution of integral equations.

In **CHAPTER – VI**, we give a new technique to compute coupled coincidence points in various spaces. Also, we rectify some errors present in the recent papers on coupled coincidence and coupled common fixed points in some spaces. This chapter has eight sections. Section 6.1 gives a brief introduction to some previous results. In section 6.2, we discuss a new technique to compute coupled coincidence points. The technique discussed in this section improves a recent technique present in the literature. In section 6.3, using the technique given in section 6.2, we improve some recent coupled coincidence point results in POMS. Section 6.4 consists of the generalization of a recent coupled coincidence point result for probabilistic φ -contraction in POMPMS by using the technique given in section 6.2. In section 6.5, using the technique given in section 6.2, we generalize a result in POGMS. Section

6.6 consists of some remarks on some recent papers concerning coupled coincidence points. In section 6.7, we point out and rectify an error in a recent paper on probabilistic φ – contraction in PGM-spaces. In section 6.8, we point out and rectify some errors in a recent paper on weakly related mappings in POMS.

In **CHAPTER – VII**, we prove some fixed point and coupled fixed point results in POMS. The results obtained are generalizations of a number of existing works. This chapter consists of four sections. Section 7.1 gives the introduction to some already existing contractions in POMS. In section 7.2, we prove some fixed point results for generalized weak $(\psi > \phi)$ – contraction mappings in POMS. Section 7.3 consists of the application of the results established in section 7.2 to coupled fixed point results. In section 7.4, we establish some coupled coincidence point and coupled common fixed point results for the pair of mappings lacking MgMP.

In **CHAPTER – VIII**, we discuss some results for w-compatible (weakly compatible) mappings, variants of weakly commuting and compatible mappings, mappings with property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property in context of coupled fixed point theory. This chapter deals with results in fuzzy metric spaces with some corresponding results in metric spaces. This chapter has five sections. Section 8.1 constitutes the introductory part. In section 8.2, we discuss variants of weakly commuting and compatible mappings in coupled fixed point theory in fuzzy metric spaces and metric spaces. Section 8.3 consists of coupled fixed point results for weakly compatible mappings, variants of weakly commuting and compatible mappings in fuzzy metric spaces. In section 8.4, we study the notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property and utilize these notions to generalize some existing results in coupled fixed point theory in fuzzy metric spaces. Section 8.5 is the application part which consists of the metrical version of some results proved in fuzzy metric spaces in the earlier sections of this chapter.

In the last, the presented work is culminated with conclusion and scope for further work.

CHAPTER – II

LITERATURE REVIEW

2.1. METRICAL SURVEY OF FIXED POINT AND COUPLED FIXED POINT THEORY

The notion of metric space was introduced by M. Frechet (1878-1973) in his doctoral thesis and this notion plays an important role in the study of topology and functional analysis.

Now-a-days, the study of metrical fixed point theory is receiving great attention of researchers due to its broad area of applications in various disciplines. An early fixed point result in topology was formulated by Brouwer [31] in 1912, which states, "Any continuous function from the closed unit ball in n-dimensional Euclidean space to itself must have a fixed point". This result was further extended by Schauder [32] in 1930 to closed, bounded and convex subsets of Banach spaces.

On the other hand, in 1922, S. Banach [33] gave one of the most important fixed point theorem, famously known as Banach fixed point theorem or Banach contraction principle (**BCP**). A self mapping ħ defined on a metric space (Ҳ, ḑ) is called a **contraction mapping** if

$$ḑ(ħϰ, ħy) \leq ķ\, ḑ(ϰ, y), \text{ for all } ϰ, y \in Ҳ \text{ and } 0 \leq ķ < 1.$$

BCP states, "Every contraction mapping on a complete metric space has a unique fixed point". This contraction principle has many applications which are scattered throughout in almost all the branches of mathematics. BCP has been enjoyed and extended by various authors over the years in different directions. In 1969, an important generalization of BCP was formulated by Boyd and Wong [34], by considering a non-linear contraction of the form:

$$ḑ(ħϰ, ħy) \leq \psi\big(ḑ(ϰ, y)\big), \text{ where } \psi \text{ being some appropriate function on } \mathbb{R}^+.$$

Following the view point of Boyd and Wong [34], different authors generalized and extended BCP by considering different assumptions on ψ. This was the beginning of a new era of functions which are now-a-days popularly known as comparison functions. In connection with the function ψ: $\mathbb{R}^+ \to \mathbb{R}^+$, different authors have considered some of the following properties:

(cf-i) ψ is non-decreasing;

(cf-ii) ψ(ƭ) < ƭ for all ƭ > 0;

(cf-iii) $\psi(0) = 0$;

(cf-iv) ψ is continuous;

(cf-v) $\lim_{n\to\infty} \psi^n(\mathfrak{t}) = 0$ for all $\mathfrak{t} \geq 0$;

(cf-vi) $\sum_{n=0}^{\infty} \psi^n(\mathfrak{t})$ converges for all $\mathfrak{t} > 0$, ψ^n is the nth iterate of ψ;

(cf-vii) $\psi(\mathfrak{t}) = 0$ iff $\mathfrak{t} = 0$;

(cf-viii) $\psi(\mathfrak{t}) > 0$ for $\mathfrak{t} \in \mathbb{R}^+\backslash\{0\}$;

(cf-ix) $\lim_{\mathfrak{r} \to \mathfrak{t}^+} \psi(\mathfrak{r}) < \mathfrak{t}$ for each $\mathfrak{t} > 0$;

(cf-x) $\lim_{\mathfrak{t}\to\infty} \psi(\mathfrak{t}) = \infty$;

(cf-xi) ψ is lower semi-continuous.

Clearly, it follows that

(cf-i) and (cf-ii) implies (cf-iii);

(cf-ii) and (cf-iv) implies (cf-iii);

(cf-i) and (cf-v) implies (cf-ii).

A function ψ satisfying (cf-i) and (cf-v), that is, ψ is non-decreasing and $\lim_{n\to\infty} \psi^n(\mathfrak{t}) = 0$ for all $\mathfrak{t} \geq 0$ is said to be a **comparison function**. In the present work, we denote by CF-Ψ, the family of all comparison functions.

A function ψ satisfying (cf-i) and (cf-vi), that is, ψ is non-decreasing and $\sum_{n=0}^{\infty} \psi^n(\mathfrak{t})$ converges for all $\mathfrak{t} > 0$ is said to be a **(c)-comparison function**. In the present work, we denote by CCF-Ψ, the family of all (c)-comparison functions.

The study of these functions has been carried out by various authors (see [35], [36], [37]). Clearly, "any (c)-comparison function is a comparison function" and "any comparison function satisfies (cf-iii)". Different authors modified these comparison functions as per the requirement of their work.

In 1969, Meir and Keeler [38] generalized BCP by using a strict contraction condition which after their name is popularly known as Meir-Keeler contraction.

Theorem 2.1.1 ([38]). Let $(\mathfrak{X}, \mathfrak{d})$ be a complete metric space and $\hbar: \mathfrak{X} \to \mathfrak{X}$ be a given mapping. Suppose for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \mathfrak{d}(\varkappa, y) < \varepsilon + \delta(\varepsilon) \Longrightarrow \mathfrak{d}(\hbar\varkappa, \hbar y) < \varepsilon, \tag{2.1.1}$$

for all $\varkappa$, $y \in \mathfrak{X}$. Then, $\hbar$ has a unique fixed point $\varkappa_0 \in \mathfrak{X}$ and for all $\varkappa \in \mathfrak{X}$, the sequence $\{\hbar^n\varkappa\}$ converges to $\varkappa_0$.

In 1973, Geraghty [39] gave an interesting generalization of BCP using the class $\mathfrak{R}$ of the functions $\beta\colon \mathbb{R}^{+} \to [0, 1)$ satisfying the condition:

$$\beta(\mathfrak{t}_n) \to 1 \text{ implies } \mathfrak{t}_n \to 0.$$

Theorem 2.1.2 ([39]). Let $(\mathfrak{X}, \mathfrak{d})$ be a complete metric space and $\hbar$ be a self-mapping on $\mathfrak{X}$ such that there exists $\beta \in \mathfrak{R}$ satisfying

$$\mathfrak{d}(\hbar\varkappa, \hbar y) \leq \beta\big(\mathfrak{d}(\varkappa, y)\big)\mathfrak{d}(\varkappa, y), \tag{2.1.2}$$

for all $\varkappa$, $y \in \mathfrak{X}$. Then, the sequence $\{\varkappa_n\}$ defined by $\varkappa_n = \hbar\varkappa_{n-1}$ for each $n \geq 1$ converges to the unique fixed point of $\hbar$ in $\mathfrak{X}$.

Recently, the theory of fixed points has been receiving much attention in POMS. Ran and Reurings [40] established an analogue of BCP in POMS. The significant feature of the work produced in [40] was that the contractive condition on the nonlinear map was assumed to hold only for the elements that were comparable w.r.t. partial order. Further, in [40], the authors assumed the following assumption on the poset $(\mathfrak{X}, \preccurlyeq)$:

Assumption 2.1.1 ([40]). $\mathfrak{X}$ has the property: "every pair $\varkappa$, $y \in \mathfrak{X}$ has a lower bound and an upper bound".

In the present study, a **partially ordered complete metric space** (**POCMS**) refers to the complete metric space endowed with a partial order. In particular, $(\mathfrak{X}, \preccurlyeq, \mathfrak{d})$ is called a **POCMS**, if $\mathfrak{X}$ is a non-empty set such that:

(i) $(\mathfrak{X}, \preccurlyeq)$ is a poset;

(ii) $\mathfrak{d}$ is a metric on $\mathfrak{X}$ such that $(\mathfrak{X}, \mathfrak{d})$ is a complete metric space.

Recall that, a partially ordered metric space (POMS) refers to the metric space endowed with a partial order.

Following is the main result in [40]:

Theorem 2.1.3 ([40]). Let $(\mathfrak{X}, \preccurlyeq, \mathfrak{d})$ be a POCMS with Assumption 2.1.1. If $\hbar$ is a monotone and continuous self-mapping on $\mathfrak{X}$ and there exists k, $0 < k < 1$ such that

$$\mathfrak{d}\big(\hbar(\varkappa), \hbar(y)\big) \leq k\, \mathfrak{d}(\varkappa, y), \tag{2.1.3}$$

for $\varkappa \succcurlyeq y$. If there exists $\varkappa_0 \in \mathfrak{X}$ such that

$$\varkappa_0 \preccurlyeq \hbar(\varkappa_0) \text{ or } \varkappa_0 \succcurlyeq \hbar(\varkappa_0), \tag{2.1.4}$$

then, $\hbar$ has a unique fixed point $\bar{\varkappa}$. Moreover, for every $\varkappa \in \mathfrak{X}$, $\lim_{n\to\infty} \hbar^{n}(\varkappa) = \bar{\varkappa}$.

Nieto and L$\acute{o}$pez [41] extended the results of Ran and Reurings [40]. In [41], authors presented an extension of BCP in POMS that permits to consider the discontinuous functions also. Further, Nieto and L$\acute{o}$pez [41] also proved the existence of solution for the following first-order periodic problem:

$$\left.\begin{array}{c} \nu'(\mathfrak{t}) = \mathrm{K}(\mathfrak{t}, \nu(\mathfrak{t})),\ \mathfrak{t} \in \mathrm{I} = [0,\mathrm{T}] \\ \nu(0) = \nu(\mathrm{T}), \end{array}\right\} \tag{2.1.5}$$

where T > 0, and K: I × ℝ → ℝ is a continuous function.

The main result given by Nieto and L$\acute{o}$pez [41] is as follows:

Theorem 2.1.4 ([41]). Let ($\mathfrak{X}$, $\preccurlyeq$, $\mathfrak{d}$) be a POCMS. Let $\hbar$ be a non-decreasing and continuous self-mapping on $\mathfrak{X}$ such that there exists $\mathrm{k} \in [0, 1)$ with

$$\mathfrak{d}(\hbar\varkappa, \hbar y) \leq \mathrm{k}\mathfrak{d}(\varkappa, y), \text{ for all } \varkappa \succcurlyeq y. \tag{2.1.6}$$

If there exists $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar(\varkappa_0)$, then $\hbar$ has a fixed point.

Interestingly, Theorem 2.1.4 still holds if the continuity hypothesis of $\hbar$ is replaced by the following assumption on $\mathfrak{X}$:

Assumption 2.1.2 ([41]). $\mathfrak{X}$ has the property: "if a non-decreasing sequence $\{\varkappa_n\} \to \varkappa$, then $\varkappa_n \preccurlyeq \varkappa$ for all n".

In [40], authors proved the uniqueness of fixed point by considering Assumption 2.1.1. In [41], it was asserted that the uniqueness of fixed point can be achieved by considering the following hypothesis which is weaker than Assumption 2.1.1.

Assumption 2.1.3 ([41]). $\mathfrak{X}$ has the property: "every pair of elements has a lower bound or an upper bound".

Also, in [41] it was asserted that Assumption 2.1.3 is equivalent to the following assumption:

Assumption 2.1.4 ([41]). $\mathfrak{X}$ has the property: "for every $\varkappa$, y ∈ $\mathfrak{X}$, there exists $\mathfrak{z} \in \mathfrak{X}$ which is comparable to $\varkappa$ and y".

A slight modification of the results proved in [41] was produced in [42] by considering the following assumption:

Assumption 2.1.5 ([42]). $\mathfrak{X}$ has the property: "if $\{\varkappa_n\} \to \varkappa$ is a sequence in $\mathfrak{X}$ whose consecutive terms are comparable, then there exists a subsequence $\{\varkappa_{n_k}\}_{k\in\mathbb{N}}$ of $\{\varkappa_n\}_{n\in\mathbb{N}}$ such that every term is comparable to the limit $\varkappa$".

Using Assumption 2.1.5, authors in [42] proved the following result:

Theorem 2.1.5 ([42]). Let ($\mathfrak{X}$, $\preccurlyeq$, $\mathfrak{d}$) be a POCMS with Assumption 2.1.3. Let $\hbar$ be a non-increasing self mapping on $\mathfrak{X}$ such that there exists $\mathrm{k} \in [0, 1)$ satisfying (2.1.6). Suppose either

(a) $\hbar$ is continuous or (b) $\mathfrak{X}$ assumes Assumption 2.1.5.

If there exists $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar(\varkappa_0)$ or $\varkappa_0 \succcurlyeq \hbar(\varkappa_0)$, then $\hbar$ has a unique fixed point.

Generalizing BCP has always been a heavily investigated research branch. Weak and generalized contractions are the generalizations of the Banach contraction mapping, which have been studied over the years by various authors using the altering distance functions. These functions are sometimes referred as control functions, basically introduced by Khan et al. [43] in 1984.

Definition 2.1.1 ([43]). An **altering distance function** is a function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies (cf-i), (cf-iv) and (cf-vii), that is

(ψ_{i}) ψ is non-decreasing and continuous;

(ψ_{ii}) $\psi(\mathfrak{t}) = 0$ iff $\mathfrak{t} = 0$.

For brevity, we call altering distance function as **ADF**.

Alber and Guerre-Delabriere [44], studied weak contractions in Hilbert spaces and proved the existence of fixed points therein. Later on, Rhoades [45] utilized weak contractions in complete metric spaces and proved the following result:

Theorem 2.1.6 ([45]). Let $\hbar$ be a self-mapping defined on a complete metric space $(\mathfrak{X}, \mathfrak{d})$ satisfying for all $\varkappa, \mathrm{y} \in \mathfrak{X}$, the following condition:

$$\mathfrak{d}(\hbar\varkappa, \hbar\mathrm{y}) \leq \mathfrak{d}(\varkappa, \mathrm{y}) - \psi\big(\mathfrak{d}(\varkappa, \mathrm{y})\big), \tag{2.1.7}$$

where $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing function satisfying (cf-iii). Then, $\hbar$ has a unique fixed point in $\mathfrak{X}$.

In 2008, Dutta and Choudhury [46] generalized Theorem 2.1.6 under a more generalized contraction and proved the following interesting result:

Theorem 2.1.7 ([46]). Let $(\mathfrak{X}, \mathfrak{d})$ be a complete metric space and $\hbar$ be a self mapping on $\mathfrak{X}$ satisfying

$$\psi\big(\mathfrak{d}(\hbar\varkappa, \hbar\mathrm{y})\big) \leq \psi\big(\mathfrak{d}(\varkappa, \mathrm{y})\big) - \phi\big(\mathfrak{d}(\varkappa, \mathrm{y})\big), \tag{2.1.8}$$

where ψ and ϕ are ADF. Then, $\hbar$ has a unique fixed point.

On the other hand, Harjani and Sadarangani [47, 48] investigated fixed points for weak and generalized contractions in the metric spaces endowed with a partial order by using the ADF.

Theorem 2.1.8 ([47]). Let $(\mathfrak{X}, \preccurlyeq, \mathfrak{d})$ be a POCMS and $\hbar$ be a non-decreasing and continuous self-mapping on $\mathfrak{X}$ such that

$$\mathfrak{d}(\hbar\varkappa, \hbar\mathrm{y}) \leq \mathfrak{d}(\varkappa, \mathrm{y}) - \psi\big(\mathfrak{d}(\varkappa, \mathrm{y})\big), \tag{2.1.9}$$

for all $\varkappa$, y ∈ $\mathfrak{X}$ with $\varkappa \succcurlyeq$ y, where ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying (cf-i), (cf-iii), (cf-iv), (cf-viii) and (cf-x). If there exists $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ has a fixed point.

Theorem 2.1.9 ([48]). Let ($\mathfrak{X}$, ≼, ḑ) be a POCMS and ħ be a non-decreasing and continuous self-mapping on $\mathfrak{X}$ such that

$$\psi\big(d(\hbar\varkappa, \hbar y)\big) \leq \psi\big(d(\varkappa, y)\big) - \phi\big(d(\varkappa, y)\big), \qquad (2.1.10)$$

for all $\varkappa$, y ∈ $\mathfrak{X}$ with $\varkappa \succcurlyeq$ y, where ψ and ϕ are ADF. If there exists $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ has a fixed point.

An important category in fixed point theory is the family of problems dealing with common fixed points. In 1976, Jungck [49] formulated a common fixed point result using the concept of commuting mappings.

Let ($\mathfrak{X}$, ḑ) be a metric space, then, we have the following definitions due to Jungck [49]:

Definition 2.1.2 ([49]). (i) For the self mappings ħ and g defined on $\mathfrak{X}$, an element ɑ ∈ $\mathfrak{X}$ is called **coincidence point** of ħ and g if ħɑ = gɑ and **common fixed point** of ħ and g if ħɑ = gɑ = ɑ.

(ii) The mappings ħ, g: $\mathfrak{X} \to \mathfrak{X}$ are said to be **commuting** if ħg$\varkappa$ = għ$\varkappa$, for all $\varkappa \in \mathfrak{X}$. In this case, we say that the mappings ħ, g **commutes** and the pair (ħ, g) is **commuting**.

The concept of commuting maps has been generalized by different authors in many ways. An important generalization of this notion was introduced in [50], known as “weak compatibility”.

Definition 2.1.3 ([50]). Two maps ħ, g: $\mathfrak{X} \to \mathfrak{X}$ are said to be **weakly compatible** (**weak compatible**) if ħg$\varkappa$ = għ$\varkappa$, whenever ħ$\varkappa$= g$\varkappa$, where $\varkappa \in \mathfrak{X}$.
In this case, we say that the pair (ħ, g) is **weak compatible** or **weakly compatible**.

In their remarkable work, Agarwal et al. [51] presented some new results for generalized nonlinear contractions. Results proved in [51] are given below:

Theorem 2.1.10 ([51]). Let ($\mathfrak{X}$, ≼, ḑ) be a POCMS. Assume there is a function ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying (cf-i) and (cf-v). Also, suppose that ħ is a non-decreasing self-mapping on $\mathfrak{X}$ with

$$d(\hbar\varkappa, \hbar y) \leq \psi\left(\max\left\{d(\varkappa, y), d(\varkappa, \hbar\varkappa), d(y, \hbar y), \frac{1}{2}[d(\varkappa, \hbar y) + d(y, \hbar\varkappa)]\right\}\right), \quad (2.1.11)$$

for all $\varkappa \succcurlyeq$ y. Also, suppose either

(a) ħ is continuous or (b) $\mathfrak{X}$ assumes Assumption 2.1.2.

If there exists an $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ has a fixed point.

Theorem 2.1.11 ([51]). Let ($\mathfrak{X}$, $\preccurlyeq$, $\mathfrak{d}$) be a POCMS. Assume there exists a function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (cf-ii) and (cf-iv) and also suppose ħ is a non-decreasing self-mapping on $\mathfrak{X}$ with

$$\mathfrak{d}(\hbar\varkappa, \hbar y) \leq \psi(\max\{\mathfrak{d}(\varkappa, y), \mathfrak{d}(\varkappa, \hbar\varkappa), \mathfrak{d}(y, \hbar y)\}), \qquad (2.1.12)$$

for all $\varkappa \succcurlyeq y$. Also, suppose either

(a) ħ is continuous or (b) $\mathfrak{X}$ assumes Assumption 2.1.2.

If there exists an $\varkappa_0 \in \mathfrak{X}$ such that $\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ has a fixed point.

Recently, Ćirić et al. [52] introduced the notion of g-monotone mapping and utilized it to prove a common fixed point result for generalized nonlinear contractions.

Definition 2.1.4 ([52]). Suppose ($\mathfrak{X}$, $\preccurlyeq$) is a poset and ħ, g are self-mappings on $\mathfrak{X}$. Then ħ is said to be **g-non-decreasing** if for $\varkappa$, y ∈ $\mathfrak{X}$,

$$g\varkappa \preccurlyeq gy \text{ implies } \hbar\varkappa \preccurlyeq \hbar y.$$

If g is the identity map on $\mathfrak{X}$, then ħ is said to be a **non-decreasing mapping**.

In their work, Ćirić et al. [52] assumed the following assumption on $\mathfrak{X}$:

Assumption 2.1.6 ([52]). $\mathfrak{X}$ has the property: "if $\{g\varkappa_n\} \subset \mathfrak{X}$ is a non-decreasing sequence with $g\varkappa_n \to g\zeta$ in g($\mathfrak{X}$), then $g\varkappa_n \preccurlyeq g\zeta$, $g\zeta \preccurlyeq gg\zeta$ for all n ∈ ℕ hold".

The main result given by Ćirić et al. [52] is as follows:

Theorem 2.1.12 ([52]). Let ($\mathfrak{X}$, $\preccurlyeq$, $\mathfrak{d}$) be a POCMS. Assume there is a function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (cf-ii) amd (cf-iv). Suppose that ħ, g be two self-mappings on $\mathfrak{X}$ such that ħ($\mathfrak{X}$) ⊆ g($\mathfrak{X}$), ħ is g-non-decreasing and

$$\mathfrak{d}(\hbar\varkappa, \hbar y) \leq \max\left\{\begin{array}{c}\varphi\big(\mathfrak{d}(g\varkappa, gy)\big), \varphi\big(\mathfrak{d}(g\varkappa, \hbar\varkappa)\big), \varphi\big(\mathfrak{d}(gy, \hbar y)\big), \\ \varphi\left(\frac{\mathfrak{d}(g\varkappa, \hbar y)+\mathfrak{d}(gy, \hbar\varkappa)}{2}\right)\end{array}\right\}, \qquad (2.1.13)$$

for all $\varkappa$, y ∈ $\mathfrak{X}$ with $g\varkappa \succcurlyeq gy$. Also suppose $\mathfrak{X}$ assumes Assumption 2.1.6 and g($\mathfrak{X}$) is closed. If there exists $\varkappa_0 \in \mathfrak{X}$ with $g\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ and g have a coincidence. Further, if ħ and g commutes at their coincidence points, then ħ and g have a common fixed point.

On the other hand, Amini-Harandi and Emami [53] extended Geraghty's result (see, Theorem 2.1.2) in the setting of metric spaces endowed with a partial order as follows:

Theorem 2.1.13 ([53]). Let ($\mathfrak{X}$, $\preccurlyeq$, $\mathfrak{d}$) be a POCMS, ħ be an increasing self-mapping on $\mathfrak{X}$ and there exists an element $\varkappa_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \hbar\varkappa_0$. Suppose that there exists $\beta \in \mathfrak{R}$ such that the mapping ħ satisfy (2.1.2) for all $\varkappa$, y ∈ $\mathfrak{X}$ with $\varkappa \succcurlyeq y$. Assume either

(a) ħ is continuous or (b) Ꭓ assumes Assumption 2.1.2.

Besides, if for each $\varkappa$, y ∈ Ꭓ, there exists ʐ ∈ Ꭓ comparable to $\varkappa$ and y, then ħ has a unique fixed point.

Now-a-days, authors are taking keen interest to extend, generalize and unify fixed point results for the self-mappings on uni-dimensional space Ꭓ to the results for the mappings having Ꭓ × Ꭓ as domain, whereas, Ꭓ being the co-domain. The first attempt in this direction was made by Guo and Lakshmikantham [54] but the line of research in this direction grew rapidly with the worth mentioning work of Bhaskar and Lakshmikantham [55], where they introduced the mixed monotone property and proved some coupled fixed point results for mappings with this property.

Definition 2.1.5 ([54, 55]). An element ($\varkappa$, y) ∈ Ꭓ × Ꭓ is called a **coupled fixed point** of the mapping Ꞙ: Ꭓ × Ꭓ → Ꭓ if Ꞙ($\varkappa$, y) = $\varkappa$ and Ꞙ(y, $\varkappa$) = y.

Definition 2.1.6 ([55]). Let (Ꭓ, ≼) be a poset. The mapping Ꞙ: Ꭓ × Ꭓ → Ꭓ is said to have the **mixed monotone property**, if Ꞙ($\varkappa$, y) is monotone non-decreasing in $\varkappa$ and monotone non-increasing in y, that is, for any $\varkappa$, y ∈ Ꭓ,

$$\varkappa_1, \varkappa_2 \in Ꭓ, \qquad \varkappa_1 \preccurlyeq \varkappa_2 \text{ implies } Ꞙ(\varkappa_1, y) \preccurlyeq Ꞙ(\varkappa_2, y)$$

and

$$y_1, y_2 \in Ꭓ, \qquad y_1 \preccurlyeq y_2 \text{ implies } Ꞙ(\varkappa, y_1) \succcurlyeq Ꞙ(\varkappa, y_2).$$

For brevity, we call mixed monotone property as **MMP**.

If a mapping Ꞙ has MMP, then Ꞙ is said to be a **mixed monotone mapping** or **operator**.

Bhaskar and Lakshmikantham [55] proved the following result:

Theorem 2.1.14 ([55]). Let (Ꭓ, ≼, ȡ) be a POCMS and Ꞙ: Ꭓ × Ꭓ → Ꭓ be a continuous mapping having the MMP on Ꭓ. Assume there exists a ķ ∈ [0, 1) such that for $\varkappa$, y in Ꭓ with $\varkappa$ ≽ u and y ≼ v, we have

$$ȡ(Ꞙ(\varkappa, y), Ꞙ(u, v)) \leq \frac{ķ}{2}[ȡ(\varkappa, u) + ȡ(y, v)]. \tag{2.1.14}$$

Suppose that Ꭓ has the following property:

(P1) "there exist two elements $\varkappa_0$, y_0 ∈ Ꭓ with $\varkappa_0 \preccurlyeq Ꞙ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ꞙ(y_0, \varkappa_0)$",

then, Ꞙ has a coupled fixed point in Ꭓ.

It has also been shown in [55] that the continuity assumption of the mapping Ꞙ in Theorem 2.1.14 can be replaced by considering the following assumption on Ꭓ:

Assumption 2.1.7 ([55]). Ꭓ has the property:

(i) "if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset Ꭓ$ converges to $\varkappa$, then $\varkappa_n \preccurlyeq \varkappa$ for all n";

(ii) “if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset X$ converges to y, then $y \preccurlyeq y_n$ for all n”.

Likewise, in order to produce the existence of coupled coincidence points some authors including Choudhury et al. [56], Karapinar et al. [57] and others replaced the continuity hypothesis of the mapping F by the following assumption:

Assumption 2.1.8 ([56]). X has the property:

(i) “if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset X$ converges to $\varkappa$, then $g\varkappa_n \preccurlyeq g\varkappa$ for all n”;

(ii) “if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset X$ converges to y, then $gy \preccurlyeq gy_n$ for all n”.

Using ADF, Harjani et al. [58] presented the following result which extends Theorem 2.1.9 for mappings satisfying MMP:

Theorem 2.1.15 ([58]). Let $(X, \preccurlyeq, d)$ be a POCMS and $F: X \times X \to X$ be a mapping having MMP on X such that

$$\varphi\Big(d\big(F(\varkappa, y), F(u, v)\big)\Big) \leq \varphi(\max\{d(\varkappa, u), d(y, v)\}) - \phi(\max\{d(\varkappa, u), d(y, v)\}), \quad (2.1.15)$$

for all $\varkappa \succcurlyeq u$, $y \preccurlyeq v$, where φ, ϕ are ADF. Suppose either

(a) F is continuous, or (b) X assumes Assumption 2.1.7.

If X has property (P1), then F has a coupled fixed point in X.

On the other hand, Lakshmikantham and Ćirić [59] extended the notion of mixed monotone property to mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [55] for a pair of commutative maps. Since then, the concept has been of great interest for researchers.

Definition 2.1.7 ([59]). Let $(X, \preccurlyeq)$ ba a poset and $F: X \times X \to X$, $g: X \to X$ be two mappings. The mapping F is said to have the **mixed g-monotone property**, if $F(\varkappa, y)$ is monotone g-non-decreasing in $\varkappa$ and is monotone g-non-increasing in y, that is, for $\varkappa, y \in X$,

$$\varkappa_1, \varkappa_2 \in X, \quad g\varkappa_1 \preccurlyeq g\varkappa_2 \quad \text{implies} \quad F(\varkappa_1, y) \preccurlyeq F(\varkappa_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preccurlyeq gy_2 \quad \text{implies} \quad F(\varkappa, y_1) \succcurlyeq F(\varkappa, y_2).$$

For brevity, we call mixed g-monotone property as **MgMP**.

If a mapping F has MgMP, then F is said to be a **mixed g-monotone mapping** or **operator**.

Definition 2.1.8 ([59]). Let F: $X \times X \to X$ and g: $X \to X$ be two mappings, an element $(\varkappa, y) \in X \times X$ is called a

(i) **coupled coincidence point** of F and g, if $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

(ii) **coupled common fixed point** of F and g, if $F(\varkappa, y) = g\varkappa = \varkappa$ and $F(y, \varkappa) = gy = y$.

If $(\varkappa, y) \in X \times X$ is a coupled coincidence point of F and g, then $(g\varkappa, gy)$ is said to be **point of coupled coincidence** of F and g.

Further, if $\varkappa \in X$ be such that $F(\varkappa, \varkappa) = g\varkappa = \varkappa$, then $\varkappa$ is called a **common fixed point** of F and g.

Definition 2.1.9 ([59]). The mappings F: $X \times X \to X$ and g: $X \to X$ are said to be **commutative** if $gF(\varkappa, y) = F(g\varkappa, gy)$ for all $\varkappa, y \in X$.

In this case, we say that F, g commutes and the pair (F, g) is said to be **commutative**.

Theorem 2.1.16 ([59]). Let $(X, \preccurlyeq, d)$ be a POCMS. Assume there is a function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying (cf-ii) and (cf-ix). Also suppose that F: $X \times X \to X$ and g: $X \to X$ be two mappings such that F has MgMP and

$$d\big(F(\varkappa, y), F(u, v)\big) \leq \varphi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right), \tag{2.1.16}$$

for all $\varkappa$, y, u, v $\in X$ for which $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$. Also suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and suppose either

(a) F is continuous, or (b) X assumes Assumption 2.1.7.

Suppose that X has the following property:

(P2) "there exist two elements $\varkappa_0$, $y_0 \in X$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$".

Then, F and g have a coupled coincidence point in X.

Later on, Choudhury and Kundu [60] introduced the notion of compatible mappings in coupled fixed point theory and utilized the notion to improve the results of Lakshmikantham and Ćirić [59].

Definition 2.1.10 ([60]). The mappings F: $X \times X \to X$ and g: $X \to X$ are said to be **compatible** if

$$\lim_{n\to\infty} d(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n)) = 0, \ \lim_{n\to\infty} d(gF(y_n, \varkappa_n), F(gy_n, g\varkappa_n)) = 0,$$

whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$ and $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, y $\in X$.

In this case, we say that the pair (F, g) is **compatible**.

Theorem 2.1.17 ([60]). Let (Ҳ, ≼, ḑ) be a POCMS. Assume there is a function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying (cf-ii) and (cf-ix). Also suppose that Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ be two mappings with Ғ having MgMP and satisfy (2.1.16) for all ϰ, y, u, v ∈ Ҳ with gϰ ≼ gu and gy ≽ gv. Further, suppose Ғ(Ҳ × Ҳ) ⊆ g(Ҳ), the pair (Ғ, g) is compatible, g is continuous and monotone increasing. Also, suppose either

(a) Ғ is continuous, or (b) Ҳ assumes Assumption 2.1.7.

If Ҳ has the property (P2), then Ғ and g have a coupled coincidence point in Ҳ.

In their nice work, Choudhury et al. [56] extended the work of Bhaskar and Lakshmikantham [55] for a pair of compatible mappings and improved the results of Harjani et al. [58] under the following result:

Theorem 2.1.18 ([56]). Let (Ҳ, ≼, ḑ) be a POCMS. Let ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying (cf-iv) and (cf-vii) and ψ be an ADF. Let Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ be two mappings such that Ғ has MgMP on Ҳ and

$$\psi\left(ḑ\big(Ғ(ϰ, y), Ғ(u, v)\big)\right) \le \psi(\max\{ḑ(gϰ, gu), ḑ(gy, gv)\}) - \phi(\max\{ḑ(gϰ, gu), ḑ(gy, gv)\}), \qquad (2.1.17)$$

for all ϰ, y, u, v ∈ Ҳ for which gϰ ≽ gu, gy ≼ gv. Suppose Ғ(Ҳ × Ҳ) ⊆ g(Ҳ), g be continuous and the pair (Ғ, g) is compatible. Suppose either

(a) Ғ is continuous, or (b) Ҳ assumes Assumption 2.1.8.

If Ҳ has the property (P2), then Ғ and g have a coupled coincidence point in Ҳ.

On the other hand, Abbas et al. [61] introduced the concept of w-compatible mappings, following which, some authors (see [62], [63]) established coupled common fixed point results for the similar notion of weakly compatible mappings.

Definition 2.1.11 (i) ([61]). The mappings Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ are called **w-compatible**, if gҒ(ϰ, y) = Ғ(gϰ, gy) whenever gϰ = Ғ(ϰ, y) and gy = Ғ(y, ϰ) for ϰ, y ∈ Ҳ.

Here, we say that the pair (Ғ, g) is **w-compatible**.

(ii) ([63]). The mappings Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ are said to be **weakly compatible** if gҒ(ϰ, y) = Ғ(gϰ, gy) and gҒ(y, ϰ) = Ғ(gy, gϰ) whenever gϰ = Ғ(ϰ, y) and gy = Ғ(y, ϰ) for ϰ, y ∈ Ҳ.

In this case, we say that the pair (Ғ, g) is called **weakly compatible**.

Interestingly, the concepts of w-compatible mappings and weakly compatible mappings are equivalent.

Remark 2.1.1. Compatible mappings are w-compatible, however, converse need not be true in general.

On the other hand, by assigning y = $\varkappa$ in the Definition 2.1.11, the concept of w^*-compatible mappings came into existence which has been enjoyed by various authors (see, [64], [65], [66]).

Definition 2.1.12 ([61, 65]). The mappings $F: X \times X \to X$ and $g: X \to X$ are called **w^*- compatible**, if $gF(\varkappa, \varkappa) = F(g\varkappa, g\varkappa)$ whenever $g\varkappa = F(\varkappa, \varkappa)$ for $\varkappa \in X$.

Then, we say that the pair (F, g) is **w^*-compatible**.

Remark 2.1.2. w^*- compatible mappings need not be compatible and w – compatible.

Luong and Thuan [67] generalized the results of Bhaskar and Lakshmikantham [55] by using the following class of functions:

Definition 2.1.13 ([67]). Let Φ_1 denote the class of functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

(φ_1) φ is continuous and non-decreasing;

(φ_2) $\varphi(t) = 0$ iff $t = 0$;

(φ_3) $\varphi(t + s) \le \varphi(t) + \varphi(s)$ for all $t, s \in \mathbb{R}^+$.

Let Φ_2 denote the class of functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy (φ_1), (φ_2), (φ_3) and the condition:

(φ_4) $\varphi(\alpha t) \le \alpha\varphi(t)$.

Definition 2.1.14 ([67]). Let Ψ denote the class of functions $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

(i_ψ) $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0+} \psi(t) = 0$.

Theorem 2.1.19 ([67]). Let $(X, \preccurlyeq, d)$ be a POCMS and $F: X \times X \to X$ be a mapping having the MMP on X. Suppose there exist $\varphi \in \Phi_1$, $\psi \in \Psi$ such that $\varkappa$, y, u, v $\in X$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we have

$$\varphi\left(d\big(F(\varkappa, y), F(u, v)\big)\right) \le \frac{1}{2}\varphi\big(d(\varkappa, u) + d(y, v)\big) - \psi\left(\frac{d(\varkappa, u) + d(y, v)}{2}\right), \qquad (2.1.18)$$

Suppose either

(a) F is continuous, or (b) X assumes Assumption 2.1.7.

If X has the property (P1), then F has a coupled fixed point in X.

Subsequently, Alotaibi and Alsulami [68] extended Theorem 2.1.19 for a pair of compatible mappings under the following result:

Theorem 2.1.20 ([68]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g: \mathfrak{X} \to \mathfrak{X}$ be two mapping such that F has the MgMP on $\mathfrak{X}$. Suppose there exist $\varphi \in \Phi_1$, $\psi \in \Psi$ such that for all $x, y, u, v \in \mathfrak{X}$ with $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$, we have

$$\varphi\left(d\big(F(x, y), F(u, v)\big)\right) \le \frac{1}{2}\varphi\big(d(gx, gu) + d(gy, gv)\big) - \psi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right), \qquad (2.1.19)$$

Suppose that $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$, g is continuous and the pair (F, g) is compatible. Also, suppose either

(a) F is continuous, or (b) $\mathfrak{X}$ assumes Assumption 2.1.7.

If $\mathfrak{X}$ has the property (P2), then, F and g have a coupled coincidence point in $\mathfrak{X}$.

Luong and Thuan [69] gave the following result:

Theorem 2.1.21 ([69]) . Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be the mapping having MMP on $\mathfrak{X}$. Suppose there exist $\alpha, \beta \ge 0$ with $\alpha + \beta < 1$ and $Ł \ge 0$ such that

$$d\big(F(x, y), F(u, v)\big) \le \alpha d(x, u) + \beta d(y, v) + Ł \min\left\{\begin{matrix} d(F(x, y), u), d(F(u, v), x), \\ d(F(x, y), x), d(F(u, v), u) \end{matrix}\right\}, \qquad (2.1.20)$$

for all $x, y, u, v \in \mathfrak{X}$ with $x \succcurlyeq u$ and $y \preccurlyeq v$. Also suppose either

(a) F is continuous, or (b) $\mathfrak{X}$ assumes Assumption 2.1.7.

If $\mathfrak{X}$ has the property (P1) then, F has a coupled fixed point in $\mathfrak{X}$.

On the other hand, Karapinar et al. [57] generalized Theorem 2.1.21 by giving the following result:

Theorem 2.1.22 ([57]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings such that F has the MgMP on $\mathfrak{X}$ and satisfies the following condition:

$$d\big(F(x, y), F(u, v)\big) \le \phi(\max\{d(gx, gu), d(gy, gv)\}) + Ł \min\left\{\begin{matrix} d(F(x, y), gu), d(F(u, v), gx), \\ d(F(x, y), gx), d(F(u, v), gu) \end{matrix}\right\}, \qquad (2.1.21)$$

for all $x, y, u, v \in \mathfrak{X}$ with $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$, where $Ł \ge 0$, $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\phi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ iff $t = 0$. Also, assume $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$, the pair (F, g) is compatible and both F, g are continuous. If $\mathfrak{X}$ has the property (P2), then, F and g have a coupled coincidence point in $\mathfrak{X}$.

Rasouli and Bahrampour [70] proved the following result which can be seen as an extension of Theorems 2.1.2 and 2.1.13 to coupled fixed point problems:

Theorem 2.1.23 ([70]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a continuous mapping having MMP on $\mathfrak{X}$ such that

$$d\big(F(\varkappa, y), F(u, v)\big) \leq \beta(\max\{d(\varkappa, u), d(y, v)\}) \max\{d(\varkappa, u), d(y, v)\}, \quad (2.1.22)$$

for all $\varkappa$, y, u, v $\in \mathfrak{X}$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, where $\beta \in \mathfrak{R}$. If $\mathfrak{X}$ has the property (P1), then F has a coupled fixed point in $\mathfrak{X}$.

The work in coupled fixed point theory in POMS is developing rapidly and motivates authors to explore more pivotal concepts that can generalize, extend and unify the already existing fundamentals in the literature.

2.2. SURVEY IN PARTIAL METRIC SPACES

In 1994, Matthews [1] introduced the concept of partial metric spaces and proved some results in it. Afterwards, various authors carried their work in these spaces and made continuous efforts to generalize the results in [1]. Works of Valero [71], Oltra and Valero [72] and Altun et al. [73] are some generalizations of the results in [1]. Subsequently, many authors studied various fixed point problems under different contractive conditions in these spaces (see [74], [75], [76]). In particular, authors are enjoying the conversion of fixed point results from the metric setup to the partial metric situation. As in the set up of metric spaces, authors are also taking interest in computing fixed points under weak and generalized contractions in partial metric spaces (see [77], [78], [79], [80]).

As already discussed in section 1.3, a partial metric on a nonempty set $\mathfrak{X}$ is a function $þ: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ satisfying axioms þ1, þ2, þ3 and þ4 and then, the pair $(\mathfrak{X}, þ)$ is the partial metric space.

It is worth mentioning that each partial metric þ on $\mathfrak{X}$ generates a T_0 topology $\tau_þ$ on $\mathfrak{X}$ for which the family of **open þ-balls** $\{B_þ(\varkappa, r): \varkappa \in \mathfrak{X},\ r > 0\}$, where $B_þ(\varkappa, r) = \{a \in \mathfrak{X}: þ(\varkappa, a) < þ(\varkappa, \varkappa) + r\}$ for all $\varkappa \in \mathfrak{X}$ and $r > 0$, is a **base**.

A sequence $\{\varkappa_n\}$ in $(\mathfrak{X}, þ)$ **converges** to a point $\varkappa \in \mathfrak{X}$ w.r.t. $\tau_þ$, if $\lim_{n\to\infty} þ(\varkappa, \varkappa_n) = þ(\varkappa, \varkappa)$. Symbolically, it is denoted by $\varkappa_n \to \varkappa$ as $n \to \infty$ or $\lim_{n\to\infty} \varkappa_n = \varkappa$.

If þ is a partial metric on $\mathfrak{X}$, then the function $þ^s: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ defined by

$$þ^s(\varkappa, y) = 2þ(\varkappa, y) - þ(\varkappa, \varkappa) - þ(y, y), \quad (2.2.1)$$

is a metric on $\mathfrak{X}$. Furthermore, $\lim_{n\to\infty} þ^s(\varkappa_n, \varkappa) = 0$ iff

$$þ(\varkappa, \varkappa) = \lim_{n\to\infty} þ(\varkappa_n, \varkappa) = \lim_{n,m\to\infty} þ(\varkappa_n, \varkappa_m).$$

Interestingly, “a limit of a sequence in a partial metric space need not be unique”.

Defintion 2.2.1 ([1]). Let (Ӿ, þ) be a partial metric space. Then

(i) a sequence $\{\varkappa_n\}$ in (Ӿ, þ) is called a **Cauchy sequence**, if $\lim_{n,m\to\infty} þ(\varkappa_n, \varkappa_m)$ exists finitely;

(ii) the space (Ӿ, þ) is said to be **complete**, if every Cauchy sequence $\{\varkappa_n\}$ in Ӿ converges w.r.t $\tau_þ$ to some point $\varkappa \in$ Ӿ such that $þ(\varkappa, \varkappa) = \lim_{n,m\to\infty} þ(\varkappa_n, \varkappa_m)$.

Lemma 2.2.1 ([1]). Let (Ӿ, þ) be a partial metric space, then

(i) $\{\varkappa_n\}$ is a Cauchy sequence in (Ӿ, þ) iff it is a Cauchy sequence in the metric space (Ӿ, $þ^s$);

(ii) the space (Ӿ, þ) is complete iff the metric space (Ӿ, $þ^s$) is complete.

Let (Ӿ, þ) be a partial metric, then υ: (Ӿ × Ӿ) × (Ӿ × Ӿ) → $\mathbb{R}^+$ defined by

$$\upsilon\big((\varkappa, y), (w, z)\big) = þ(\varkappa, w) + þ(y, z) \text{ for } (\varkappa, y), (w, z) \in Ӿ \times Ӿ,$$

is a partial metric on Ӿ × Ӿ.

Definition 2.2.2 ([81]). A mapping Ғ: Ӿ × Ӿ → Ӿ is said to be **continuous** at $(\varkappa, y) \in$ Ӿ × Ӿ, if for each ɍ > 0, there exists s > 0 such that $Ғ\Big(Ƀ_þ\big((\varkappa, y), s\big)\Big) \subseteq Ƀ_þ(Ғ(\varkappa, y), ɍ)$.

Lemma 2.2.2 ([82]). Let (Ӿ, þ) be a partial metric space. Then, the mapping Ғ: Ӿ × Ӿ → Ӿ is continuous iff given a sequence $\{(\varkappa_n, y_n)\}_{n \in \mathbb{N}}$ and $(\varkappa, y) \in$ Ӿ × Ӿ such that

$$\upsilon\big((\varkappa, y), (\varkappa, y)\big) = \lim_{n\to\infty} \upsilon\big((\varkappa, y), (\varkappa_n, y_n)\big),$$

implies $$þ\big(Ғ(\varkappa, y), Ғ(\varkappa, y)\big) = \lim_{n\to\infty} þ\big(Ғ(\varkappa, y), Ғ(\varkappa_n, y_n)\big).$$

In the present study, a **partially ordered complete partial metric space** (**POCPMS**) refers to the complete partial metric space endowed with a partial order. In particular, (Ӿ, ≼, þ) is called a **POCPMS**, if Ӿ is a non-empty set such that:

(i) (Ӿ, ≼) is a poset;

(ii) þ is a partial metric on Ӿ such that (Ӿ, þ) is a complete partial metric space.

Recall that, a partially ordered partial metric space (POPMS) refers to the partial metric space endowed with a partial order.

Recently, Aydi [83] extended Theorem 2.1.8 in POCPMS as follows:

Theorem 2.2.1 ([83]). Let (Ӽ, ≼, ϸ) be a POCPMS and ħ be a non-decreasing self-mapping on Ӽ. Suppose for y ≼ ϰ, we have

$$ϸ(ħϰ, ħy) \leq ϸ(ϰ, y) - \psi(ϸ(ϰ, y)), \qquad (2.2.2)$$

where ψ: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function satisfying (cf-i), (cf-iii), (cf-iv), (cf-viii) and (cf-x). Assume either

(a) ħ is continuous, or (b) Ӽ assumes Assumption 2.1.2.

If there exists $ϰ_0$ ∈ Ӽ such that $ϰ_0 ≼ ħϰ_0$, then ħ has a fixed point ɯ ∈ Ӽ. Moreover, ϸ(ɯ, ɯ) = 0.

Abdeljawad et al. [84] studied generalized weak ϕ - contraction in partial metric spaces. Subsequently, Abdeljawad [85] and Abbas and Nazir [86] established fixed point results for generalized weakly contractive mappings in these spaces.

As in metric spaces, the computation of coupled fixed points in the setup of partial metric spaces has attracted a great attention of researchers. Aydi [87] formulated the following result in partial metric spaces, which was originally proved in the setup of cone metric spaces by Sabetghadam et al. [88].

Theorem 2.2.2 ([87]). Let (Ӽ, ϸ) be a complete partial metric space and the mapping Ғ: Ӽ × Ӽ → Ӽ satisfies for all ϰ, y, u, v ∈ Ӽ, the following contractive condition,

$$ϸ(Ғ(ϰ, y), Ғ(u, v)) \leq ķ\, ϸ(ϰ, u) + l\, ϸ(y, v), \qquad (2.2.3)$$

where ķ, $l \geq 0$ constants with ķ $+ l < 1$. Then, Ғ has a unique coupled fixed point.

Now-a-days, authors are showing keen interest to obtain coupled fixed point results in partial metric spaces equipped with a partial order. The following coupled fixed point result has been proved by Alsulami et al. [89], which can be considered as the partial metric version of Theorem 2.1.19.

Theorem 2.2.3 ([89]). Let (Ӽ, ≼, ϸ) be a POCPMS and Ғ: Ӽ × Ӽ → Ӽ be a mapping having MMP on Ӽ. Suppose there exist $\varphi \in \Phi_2$, $\psi \in \Psi$ such that for all ϰ, y, u, v ∈ Ӽ with ϰ ≽ u and y ≼ v, we have

$$\varphi\left(ϸ\big(Ғ(ϰ, y), Ғ(u, v)\big)\right) \leq \frac{1}{2}\varphi\big(ϸ(ϰ, u) + ϸ(y, v)\big) - \psi\left(\frac{ϸ(ϰ, u) + ϸ(y, v)}{2}\right). \qquad (2.2.4)$$

Suppose either

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ӽ has the property (P1), then Ғ has a coupled fixed point in Ӽ.

Generalizing and extending the already existing results is a great priority of authors. Recently, Shatanawi et al. [81] extended and generalized the results of

Bhaskar and Lakshmikantham [55] and Harjani et al. [58] to partial metric situation. The work in partial metric spaces is developing enormously day-by-day.

2.3 SURVEY IN G-METRIC SPACES

Generalizing the structure of metric space has been always an area of great interest for researchers over the years and subsequently, authors have done much work to achieve this goal. One such example is the notion of D-metric spaces formulated by B.C. Dhage [5] in 1984.

In 2003, Mustafa and Sims [6] found that most of the assertions regarding the elementary topological properties of D-metric spaces were incorrect. This motivated Mustafa and Sims [7] to look out for some more congruous concept and consequently, they introduced the notion of G-metric spaces.

As previously discussed in section 1.4, a G-metric on a non-empty set Ӽ is a function G: $Ӽ \times Ӽ \times Ӽ \to \mathbb{R}^+$ satisfying axioms (G1), (G2), (G3), (G4), (G5) and then, the pair (Ӽ, G) is called a G-metric space. In [7], it was shown that "for any non-empty set Ӽ, it is possible to construct a G-metric from any metric on Ӽ". Further, corresponding to any metric space (Ӽ, ɖ), Mustafa and Sims [7] constructed the following G-metrics on Ӽ:

(E_s) $\quad G_s(ɖ)(\varkappa, y, z) = \frac{1}{3}[ɖ(\varkappa, y) + ɖ(y, z) + ɖ(\varkappa, z)]$,

(E_m) $\quad G_m(ɖ)(\varkappa, y, z) = \max\{ɖ(\varkappa, y), ɖ(y, z), ɖ(\varkappa, z)\}$.

In the same work [7], the authors also answered the converse problem, that for any G-metric G on Ӽ,

$(E_ɖ)$ $\quad ɖ_G(\varkappa, y) = G(\varkappa, y, y) + G(\varkappa, \varkappa, y)$,

defines a metric on Ӽ.

Mustafa and Sims [7] also defined the definition of symmetric G-metric spaces as follows:

Definition 2.3.1 ([7]). A G-metric space (Ӽ, G) is said to be **symmetric** if

(G6) $\quad G(\varkappa, y, y) = G(\varkappa, \varkappa, y)$, for all $\varkappa, y \in$ Ӽ.

Below are some important properties of a G-metric:

Proposition 2.3.1 ([7]). Let (Ӽ, G) be a G-metric space, then for any $\varkappa$, y, z, α in Ӽ, the following hold:

(1) if $G(\varkappa, y, z) = 0$, then $\varkappa = y = z$;

(2) $G(\varkappa, y, z) \leq G(\varkappa, \varkappa, y) + G(\varkappa, \varkappa, z)$;

(3) $G(\varkappa, y, y) \leq 2\, G(y, \varkappa, \varkappa)$;

(4) $G(\varkappa, y, z) \leq G(\varkappa, \alpha, z) + G(\alpha, y, z)$;

(5) $G(\varkappa, y, z) \leq \frac{2}{3}[G(\varkappa, y, \alpha) + G(\varkappa, \alpha, z) + G(\alpha, y, z)]$;

(6) $G(\varkappa, y, z) \leq G(\varkappa, \alpha, \alpha) + G(y, \alpha, \alpha) + G(z, \alpha, \alpha)$;

(7) $|G(\varkappa, y, z) - G(\varkappa, y, \alpha)| \leq \max\{G(\alpha, z, z), G(z, \alpha, \alpha)\}$;

(8) $|G(\varkappa, y, z) - G(\varkappa, y, \alpha)| \leq G(\varkappa, \alpha, z)$;

(9) $|G(\varkappa, y, z) - G(y, z, z)| \leq \max\{G(\varkappa, z, z), G(z, \varkappa, \varkappa)\}$;

(10) $|G(\varkappa, y, y) - G(y, \varkappa, \varkappa)| \leq \max\{G(y, \varkappa, \varkappa), G(\varkappa, y, y)\}$.

Definition 2.3.2 ([7]). Let (X, G) be a G-metric space, then for $\varkappa_0 \in X$, $r > 0$, the **ball** with centre $\varkappa_0$ and radius r is $B_G(\varkappa_0, r) = \{y \in X: G(\varkappa_0, y, y) < r\}$.

Proposition 2.3.2 ([7]). Let (X, G) be a G-metric space, then for any $\varkappa_0 \in X$ and $r > 0$, we have

(1) if $G(\varkappa_0, \varkappa, y) < r$, then $\varkappa, y \in B_G(\varkappa_0, r)$,

(2) if $y \in B_G(\varkappa_0, r)$, then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(\varkappa_0, r)$.

Mustafa and Sims [7], also noticed that for the G-metric space (X, G), the collection $\mathcal{B} = \{B_G(\varkappa, r): \varkappa \in X, r > 0\}$ is the **base** of the G-metric topology $\tau(G)$ on X.

Definition 2.3.3 ([7]). Let (X, G) be a G-metric space, then the sequence $\{\varkappa_n\} \subseteq X$ is **G-convergent** to $\varkappa$ if it converges to $\varkappa$ in the G-metric topology, $\tau(G)$.

Proposition 2.3.3 ([7]). Let (X, G) be a G-metric space, then for a sequence $\{\varkappa_n\} \subseteq X$ and point $\varkappa \in X$, the following are equivalent:

(1) $\{\varkappa_n\}$ is G-convergent to $\varkappa$;

(2) $d_G(\varkappa_n, \varkappa) \to 0$ as $n \to \infty$;

(3) $G(\varkappa_n, \varkappa_n, \varkappa) \to 0$ as $n \to \infty$;

(4) $G(\varkappa_n, \varkappa, \varkappa) \to 0$ as $n \to \infty$;

(5) $G(\varkappa_m, \varkappa_n, \varkappa) \to 0$ as $m, n \to \infty$.

"Clearly, if $\varkappa_n \to \varkappa$ in G-metric space (X, G), then for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $G(\varkappa, \varkappa_n, \varkappa_m) < \varepsilon$ for all $n, m \geq N$".

In [7], it was shown that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. This topology being Hausdorff, a sequence can converge at most to a point.

Definition 2.3.4 ([7]). Let (Ӽ, G), (Ӽ', G') be G -metric spaces, a function h: Ӽ → Ӽ' is **G-continuous** at a point $\varkappa_0 \in$ Ӽ, if $h^{-1}\left(B_{G'}(h\varkappa_0, r)\right) \in \tau(G)$, for all r > 0.

Further, the mapping h is G-continuous if it is G-continuous at all the points of Ӽ.

Proposition 2.3.4 ([7]). Let (Ӽ, G), (Ӽ', G') be G-metric spaces, then a function h: Ӽ → Ӽ' is G-continuous at a point ϰ ∈ Ӽ if and only if the sequence $\{h\varkappa_n\}$ is G'-convergent to hϰ whenever the sequence $\{\varkappa_n\}$ is G-convergent to ϰ.

Proposition 2.3.5 ([7]). Let (Ӽ, G) be a G-metric space, then the function G(ϰ, y, z) is jointly continuous in all three of its variables.

Definition 2.3.5 ([7]). Let (Ӽ, G) be a G-metric space, then a sequence $\{\varkappa_n\}$ in Ӽ is said to be **G-Cauchy** if for every $\varepsilon > 0$, there exists N ∈ ℕ such that $G(\varkappa_n, \varkappa_m, \varkappa_l) < \varepsilon$ for all n, m, $l \geq$ N, that is, $G(\varkappa_n, \varkappa_m, \varkappa_l) \to \infty$ as n, m, $l \to \infty$.

Proposition 2.3.6 ([7]). In a G-metric space (Ӽ, G), the following are equivalent:

(1) the sequence $\{\varkappa_n\}$ is G-Cauchy;

(2) for every $\varepsilon > 0$, there exists N ∈ ℕ such that $G(\varkappa_n, \varkappa_m, \varkappa_m) < \varepsilon$, for all n, m ≥ N;

(3) $\{\varkappa_n\}$ is a Cauchy sequence in the metric space (Ӽ, d_G).

Further, in [7], it has also been observed that "every G-convergent sequence in a G-metric space is G-Cauchy" and "if a G-Cauchy sequence in a G-metric space (Ӽ, G) contains a G-convergent subsequence, then the sequence itself is G-convergent".

Definition 2.3.6 ([7]). A G-metric space (Ӽ, G) is said to be **G-complete** if every G-Cauchy sequence in (Ӽ, G) is G-convergent in (Ӽ, G).

Proposition 2.3.7 ([7]). A G-metric space (Ӽ, G) is G-complete if and only if (Ӽ, d_G) is a complete metric space.

Computation of fixed points in G-metric spaces is an area of great interest for authors. Below is the first fixed point result in G-metric space, which was given by Mustafa [90]:

Theorem 2.3.1 ([90]). Let (Ӽ, G) be a complete G-metric space. Suppose there is k ∈ [0, 1) such that the self mapping ħ on Ӽ satisfies

$$G(ħ\varkappa, ħy, ħz) \leq k\, G(\varkappa, y, z), \tag{2.3.1}$$

for all ϰ, y, z in Ӽ. Then, ħ has a unique fixed point (say ɯ) and ħ is G-continuous at ɯ.

Theorem 2.3.1 is, in fact, the G-metric version of BCP and was further generalized by Shatanawi [91] under the following result:

Theorem 2.3.2 ([91]). Let (Ӿ, G) be a complete G-metric space. Suppose that the self mapping ħ on Ӿ satisfies

$$G(ħϰ, ħy, ħz) \leq \phi(G(ϰ, y, z)), \quad (2.3.2)$$

for all ϰ, y, z in Ӿ, where the function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (cf-i) and, it also satisfies (cf-v). Then ħ has a unique fixed point (say ư) and ħ is G-continuous at ư.

Latterly, different authors established several fixed point results under various contractive conditions in the setup of G-metric spaces (see [92], [93], [94], [95] etc.). Abbas and Rhoades [96] initiated the study of common fixed points in G-metric spaces. Shatanawi et al. [97] introduced the notions of weakly G-contractive and weakly G-contractive type mappings in these spaces. Aydi et al. [98] formulated results for weakly G-contraction mappings in G-metric spaces.

In the present study, a **partially ordered complete G-metric space** (**POCGMS**) refers to the complete G-metric space endowed with a partial order. In particular, (Ӿ, ≼, G) is called a **POCGMS**, if Ӿ is a non-empty set such that:

(i) (Ӿ, ≼) is a poset;

(ii) G is a G-metric on Ӿ such that (Ӿ, G) is a complete G-metric space.

Recall that, a partially ordered G-metric space (POGMS) refers to the G-metric space endowed with a partial order.

Recently, the popularity of fixed point results in POMS inspired researchers to carry out their work in POGMS. In particular, weak and generalized contractions have been enjoyed by a number of authors in the framework of POGMS (e.g., see [99], [100], [101]). Mustafa et al. [102] proved some coincidence point results for nonlinear generalized (ψ, φ) - weakly contractive mappings in POGMS.

Motivated by Bhaskar and Lakshmikantham [55], in their nice and elegant work, Choudhury and Maity [103] initiated the theory of coupled fixed points in the setup of G-metric spaces. In order to produce their results, Choudhury and Maity [103] gave the following definition:

Definition 2.3.7 ([103]). Let (Ӿ, G) be a G-metric space. A mapping Ꞙ: Ӿ × Ӿ → Ӿ is said to be **continuous**, if for any two G-convergent sequences $\{ϰ_n\}$ and $\{y_n\}$

converging to $\varkappa$ and y respectively, the sequence $\{F(\varkappa_n, y_n)\}$ is G-convergent to $F(\varkappa, y)$.

Following is the main result proved by Choudhury and Maity [103]:

Theorem 2.3.3 ([103]). Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$ be a continuous mapping having MMP on X. Assume there exists $k \in [0, 1)$ such that for $\varkappa$, y, z, u, v, w in X, the following holds:

$$G(F(\varkappa, y), F(u, v), F(w, z)) \leq \frac{k}{2}[G(\varkappa, u, w) + G(y, v, z)], \qquad (2.3.3)$$

for all $\varkappa \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$, where either $u \neq w$ or $v \neq z$. If X has property (P1), then, F has a coupled fixed point.

It was also shown in [103] that Theorem 2.3.3 still holds, if the continuity hypothesis of F be replaced by Assumption 2.1.7 w.r.t. convergence and ordering in $(X, \preccurlyeq, G)$. Aydi et al. [104] generalized the results of Choudhury and Maity [103] by using a pair of commutative mappings that satisfies the contraction condition analogous to the contraction (2.1.16) but in the setup of POGMS. Subsequently, many coupled common fixed point results were established by different authors in these spaces. Some of these results are extensions of the already existing results present in the metrical coupled fixed point theory. Cho et al. [105] established the existence and uniqueness of coupled common fixed points under contraction condition analogous to the contraction (2.1.17) but in the setup of G-metric spaces equipped with a partial order.

2.4. SURVEY IN MENGER PM-SPACES AND PGM-SPACES

Menger [9] pioneered the theory of PM-spaces in 1942 but the theory attracted the attention of authors after the distinguished work of Schweizer and Sklar [10, 11]. The theory of PM-spaces has been enjoyed in different directions, particularly as Wald spaces, Menger PM-spaces etc.

In 1966, Sehgal [13] initiated the study of fixed points in the setup of PM-spaces by proving the contraction mapping theorems therein. Afterwards, this area of research has further been explored by host of authors which includes Sehgal and Bharucha-Reid [106], Sherwood [107], Boscan [108], Cain and Kasriel [109], Istrătescu and Roventa [110], Istrătescu and Săcuiu [111] and others.

As discussed already in Section 1.5, a Menger PM-space is a triple (X, F, Δ), where X being a non-empty set, Δ a continuous t-norm and F is a mapping from

$\mathfrak{X} \times \mathfrak{X}$ into Λ^+ (where, Λ^+ denotes the set of all Menger distance distribution functions) such that, if $F_{\varkappa, y}$ denotes the value of F at the pair ($\varkappa$, y), the conditions (PM_1), (PM_2), (PM_3) hold.

Definition 2.4.1 ([11]). Let ($\mathfrak{X}$, F, Δ) be a Menger PM-space.

(i) A sequence $\{\varkappa_n\}$ in $\mathfrak{X}$ is said to be **convergent** to a point $\varkappa \in \mathfrak{X}$ (written as, $\varkappa_n \to \varkappa$), if for any $\mathfrak{f} > 0$ and $0 < \varepsilon < 1$, there exists $N \in \mathbb{N}$ such that $F_{\varkappa_n, \varkappa}(\mathfrak{f}) > 1 - \varepsilon$, whenever $n \geq N$;

(ii) A sequence $\{\varkappa_n\}$ in $\mathfrak{X}$ is said to be a **Cauchy sequence**, if for any $\mathfrak{f} > 0$ and $0 < \varepsilon < 1$, there exists $N \in \mathbb{N}$ such that $F_{\varkappa_n, \varkappa_m}(\mathfrak{f}) > 1 - \varepsilon$, whenever $n, m \geq N$;

(iii) ($\mathfrak{X}$, F, Δ) is said to be **complete** iff every Cauchy sequence in $\mathfrak{X}$ is convergent to a point in $\mathfrak{X}$.

Theorem 2.4.1 ([11]). If ($\mathfrak{X}$, F, Δ) is a Menger PM-space and $\{\mathfrak{a}_n\}$, $\{\mathfrak{b}_n\}$ are sequences in $\mathfrak{X}$ such that $\mathfrak{a}_n \to \mathfrak{a}$ and $\mathfrak{b}_n \to \mathfrak{b}$, then $\lim_{n\to\infty} F_{\mathfrak{a}_n, \mathfrak{b}_n}(\mathfrak{t}) = F_{\mathfrak{a}, \mathfrak{b}}(\mathfrak{t})$ for every continuity point t of $F_{\mathfrak{a}, \mathfrak{b}}$.

The following notion of contraction mappings on PM-spaces has been introduced by Sehgal [13]:

Definition 2.4.2 ([13]). Let ($\mathfrak{X}$, F, Δ) be a Menger PM-space and $\hbar: \mathfrak{X} \to \mathfrak{X}$ be an arbitrary mapping on $\mathfrak{X}$. Then $\hbar$ is called a **contraction** (or **probabilistic contraction**) if there exists $\mathfrak{k} \in (0, 1)$ such that for $\varkappa$, y in $\mathfrak{X}$ and $\mathfrak{f} > 0$, we have

$$F_{\hbar\varkappa, \hbar y}(\mathfrak{k}\mathfrak{f}) \geq F_{\varkappa, y}(\mathfrak{f}). \qquad (2.4.1)$$

Later on, the probabilistic contraction has been extended to the probabilistic ϕ – contraction as follows:

$$F_{\hbar\varkappa, \hbar y}\big(\phi(\mathfrak{f})\big) \geq F_{\varkappa, y}(\mathfrak{f}), \qquad (2.4.2)$$

where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a gauge function satisfying some appropriate conditions, which were subsequently weakened by different authors. It is worth mentioning here that, different authors obtained various interesting results for the probabilistic ϕ – contractions, where the gauge function (auxiliary function) ϕ assumes any one of the following assumptions:

(a) "$\phi(\mathfrak{f}) = \mathfrak{k}\mathfrak{f}$ for all $\mathfrak{f} > 0$, where $0 < \mathfrak{k} < 1$"; or

(b) "$\sum_{n=1}^{\infty} \phi^n(\mathfrak{f}) < \infty$ for all $\mathfrak{f} > 0$".

In order to weaken the condition (b), Ćirić [112] constructed the following condition:

(c) "$\phi(0) = 0$, $\phi(\mathfrak{t}) < \mathfrak{t}$ and $\lim_{\mathfrak{r}\to\mathfrak{t}^+}\phi(\mathfrak{r}) < \mathfrak{t}$ for all $\mathfrak{t} > 0$".

Subsequently, Jachymski [113] presented probabilistic ϕ – contraction, where ϕ satisfies the condition:

(d) "$0 < \phi(\mathfrak{t}) < \mathfrak{t}$ and $\lim_{n\to\infty}\phi^n(\mathfrak{t}) = 0$ for all $\mathfrak{t} > 0$".

Denote by Ω, the set of all functions ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (d). In order to weaken the condition (d), Fang [114] introduced the following condition:

(e) "for each $\mathfrak{t} > 0$, there exists $\mathfrak{r} \geq \mathfrak{t}$ such that $\lim_{n\to\infty}\phi^n(\mathfrak{r}) = 0$".

Denote by Ω_W, the set of all functions ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (e). Sometimes, authors also use the symbol φ instead of ϕ to denote the elements in Ω_W.

In the present study, a **partially ordered complete Menger PM-space** (**POCMPMS**) refers to the complete Menger PM-space endowed with a partial order. In particular, $(\mathfrak{X}, \preccurlyeq, F, \Delta)$ is called a **POCMPMS** if $\mathfrak{X}$ is a non-empty set such that:

(i) $(\mathfrak{X}, \preccurlyeq)$ is a poset; (ii) $(\mathfrak{X}, F, \Delta)$ is a complete Menger PM-space.

Recall that, a partially ordered Menger PM-space (**POMPMS**) refers to the Menger PM-space endowed with a partial order.

Now-a-days, researchers are paying much attention to study the fixed point results in Menger PM-spaces endowed with a partial order. Recently, Ćirić et al. [115] extended the results of Ran and Reurings [40] and Nieto and Rodriguez-Lopez [41, 42] to the wider class of contractive mappings from metric to probabilistic metric setup. In order to establish common fixed points in POCMPMS $(\mathfrak{X}, \preccurlyeq, F, \Delta)$, Ćirić et al. [115] considered the following contractive condition:

$$F_{A\varkappa,\, Ay}(k\mathfrak{t}) \geq \min\{F_{\hbar\varkappa,\, \hbar y}(\mathfrak{t}), F_{\hbar\varkappa,\, A\varkappa}(\mathfrak{t}), F_{\hbar y,\, Ay}(\mathfrak{t})\}, \tag{2.4.3}$$

for all $\varkappa$, $y \in \mathfrak{X}$ for which $\hbar\varkappa \preccurlyeq \hbar y$ and all $\mathfrak{t} > 0$, where $\hbar$ and A are self mappings on $\mathfrak{X}$ and $k \in (0, 1)$.

On the other hand, authors are promptly enjoying coupled fixed point problems in POMPMS. Ćirić et al. [116] introduced the notion of mixed monotone generalized contraction in these spaces and obtained some coupled coincidence point results therein. Later on, Wang et al. [117] proved the following coupled coincidence point result for nonlinear contractive mappings in these spaces:

Theorem 2.4.2 ([117]). Let (Ҳ, ≼, F, Δ) be a POCMPMS, where Δ is a t-norm of H-type. Let $\phi \in \Omega$ and Q: Ҳ × Ҳ → Ҳ, ħ: Ҳ → Ҳ be two mappings such that Q has MħMP on Ҳ and

$$F_{Q(\varkappa,\, y),\, Q(u,\, v)}\big(\phi(\mathfrak{t})\big) \geq \min\{F_{\hbar\varkappa,\, \hbar u}(\mathfrak{t}),\, F_{\hbar y,\, \hbar v}(\mathfrak{t})\}, \tag{2.4.4}$$

for all ŧ > 0 and all ϰ, y, u, v ∈ Ҳ for which ħu ≼ ħϰ and ħy ≼ ħv. Suppose ħ is continuous and commutes with Q and Q(Ҳ × Ҳ) ⊂ ħ(Ҳ). Also, suppose either

(a) Q is continuous, or (b) Ҳ assumes Assumption 2.1.7.

If Ҳ has the property (P2), then Q and ħ have a coupled coincidence point in Ҳ.

On the other hand, Đoric [118] formulated the following definition in Menger PM-spaces:

Definition 2.4.3 ([118]). Let (Ҳ, F, Δ) be a Menger PM-space. The mappings Q: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ are said to be **compatible** if

$$\lim_{n\to\infty} F_{gQ(\varkappa_n, y_n),\, Q(g\varkappa_n,\, gy_n)}(\mathfrak{t}) = 1, \quad \lim_{n\to\infty} F_{gQ(y_n,\, \varkappa_n),\, Q(gy_n,\, g\varkappa_n)}(\mathfrak{t}) = 1,$$

for all ŧ > 0, whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ҳ such that $\lim_{n\to\infty} Q(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Q(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y ∈ Ҳ.

Here, we say that the pair (Q, g) is **compatible**. The pair (Q, g) can also be represented by (Q: Ҳ × Ҳ → Ҳ, g: X → X).

Quite recently, using the gauge function $\varphi \in \Omega_W$, Choudhury et al. [119] obtained coupled coincidence points in POMPMS (Ҳ, ≼, F, Δ) for a pair of compatible mappings (Q: Ҳ × Ҳ → Ҳ, g: X → X) under the following φ-contraction:

$$F_{Q(\varkappa,\, y),\, Q(u,\, v)}\big(\phi(\mathfrak{t})\big) \geq \left[F_{g\varkappa,\, gu}(\mathfrak{t}) \,.\, F_{gy,\, gv}(\mathfrak{t})\right]^{\frac{1}{2}}, \tag{2.4.5}$$

for all ŧ > 0, ϰ, y, u, v ∈ Ҳ with gϰ ≼ gu and gy ≽ gv.

Generalizing and extending already existing notions has always been a great preeminence for researchers. Recently, Zhou et al. [15] formulated the probabilistic version of G-metric spaces which is famously known as Menger probabilistic G-metric space (**PGM-space**). PGM-space is a generalization of Menger PM-space.

As already discussed in Section 1.5, a PGM-space is a triple (Ҳ, G^*, Δ), where Ҳ is a non-empty set, Δ is a continuous t-norm and G^* is a mapping from Ҳ × Ҳ × Ҳ into Λ^+ ($G^*_{\varkappa,\, y,\, z}$ denote the value of G^* at the point (ϰ, y, z)) satisfying the conditions (PGM-1), (PGM-2), (PGM-3), (PGM-4).

Zhou et al. [15] also investigated some topological properties of PGM-space. Further, in the same work [15], some fixed point results were also established. These results were actually the probabilistic version of the already existing results, particularly, of BCP. This attracted researchers to work in the PGM-spaces also. In particular, coupled fixed point results are being enjoyed by researchers in these spaces.

Recently, Zhu et al. [120] proved their results in PGM-spaces using the following contractions:

(i) $$G^{*}_{T(\varkappa,\, y),T(a,\, q),T(h,\, l)}\big(\varphi(\mathfrak{t})\big) \geq \Big[\Delta\Big(G^{*}_{A\varkappa,\, Aa,\, Ah}(\mathfrak{t}),\, G^{*}_{Ay,\, Aq,\, Al}(\mathfrak{t})\Big)\Big]^{\frac{1}{2}}, \qquad (2.4.6)$$

where φ: $\mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ being a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{m=1}^{\infty} \varphi^{m}(\mathfrak{t}) < \infty$ for any $\mathfrak{t} > 0$ and T: $X \times X \rightarrow X$ and A: $X \rightarrow X$ be two mappings.

(ii) $$G^{*}_{T(\varkappa,\, y),T(a,\, q),T(h,\, l)}\big(\varphi(\mathfrak{t})\big) \geq \big[G^{*}_{A\varkappa,\, Aa,\, Ah}(\mathfrak{t})\,.\,G^{*}_{Ay,\, Aq,\, Al}(\mathfrak{t})\big]^{\frac{1}{2}}, \qquad (2.4.7)$$

where φ: $\mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ being a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(\mathfrak{t}) < \mathfrak{t}$ and $\lim_{m\rightarrow\infty} \varphi^{m}(\mathfrak{t}) = 0$ for any $\mathfrak{t} > 0$.

The theory of coupled fixed points in Menger PM-spaces and PGM-spaces is a dynamic study and is expanding day-by-day.

2.5. SURVEY IN FUZZY METRIC SPACES

In 1965, the introduction of fuzzy sets by Zadeh [16] proved a turning point in the field of mathematical sciences. In 1975, Kramosil and Michalek [20] laid the foundation of KM-fuzzy metric space (**KMFMS**). Grabiec [121] presented the fuzzy version of BCP in these spaces. On the other hand, Fang [122] proved some fixed point theorems for contractive type mappings in such spaces, wherein he generalized and improved the works of Edelstein [123], Istrătescu [124], Sehgal and Bharucha-Reid [106]. The result of Grabiec [121] was generalized by Subrahmanyam [125] for the pair of commuting mappings. In fact, Subrahmanyam [125] presented the fuzzy analogue of Jungck’s result [49] and therein, proposed the applicability of his result for compatible mappings. Originally, the notion of compatible mappings was framed by Jungck [126] in metric spaces which was carried in the setup of fuzzy metric spaces by Mishra et al. [127]. Vasuki [128] defined the notions of weakly commuting and R-weakly commuting maps in fuzzy metric spaces to obtain common fixed points in these spaces. Later on, the variants of compatible and weakly commuting mappings have been enjoyed by different authors to develop the common fixed point results.

On the other hand, with an aspect to access Hausdorff topology in the fuzzy metric spaces, George and Veeramani [21, 22] modified the concept of fuzzy metric spaces due to Kramosil and Michalek [20]. Afterwards, the theory of fixed points developed considerably in these spaces. Different authors established various fixed point results in fuzzy metric spaces in the sense of George and Veeramani [21, 22] (**GVFMS**).

Section 1.5 discussed the notion of t-norm while section 1.6 provided an introduction of KMFMS and GVFMS. A KMFMS is a triple $(\mathfrak{X}, \mathcal{M}, *)$, where $\mathfrak{X}$ is a nonempty set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathfrak{X}^2 \times \mathbb{R}^+$ satisfying the axioms (KM-1), (KM-2), (KM-3), (KM-4), (KM-5). On the other hand, a GVFMS is a triple $(\mathfrak{X}, \mathcal{M}, *)$, where $\mathfrak{X}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathfrak{X}^2 \times \mathbb{R}^+\backslash\{0\}$ satisfying the axioms (FM-1), (FM-2), (FM-3), (FM-4), (FM-5).

The concepts of Cauchy sequences and convergent sequences in KMFMS were defined by Grabiec [121] as follows:

Definition 2.5.1 ([121]). Let $(\mathfrak{X}, \mathcal{M}, *)$ be a fuzzy metric space, then

(i) A sequence $\{\varkappa_n\}$ in $\mathfrak{X}$ is said to be **Cauchy** if $\lim_{n\to\infty} \mathcal{M}(\varkappa_{n+p}, \varkappa_n, \mathfrak{t}) = 1$, for each $\mathfrak{t} > 0$ and $p > 0$;

(ii) A sequence $\{\varkappa_n\}$ in $\mathfrak{X}$ is **convergent** to $\varkappa \in \mathfrak{X}$ if $\lim_{n\to\infty} \mathcal{M}(\varkappa_n, \varkappa, \mathfrak{t}) = 1$, for each $\mathfrak{t} > 0$. In notations, we write $\lim_{n\to\infty} \varkappa_n = \varkappa$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is called **complete**.

Grabiec [121] also suggested that since $*$ is continuous, the limit in the above definition of convergence is uniquely determined. In the same paper [121], fuzzy version of BCP was also suggested.

Theorem 2.5.1 ([121]). Let $(\mathfrak{X}, \mathcal{M}, *)$ be a complete fuzzy metric space with (FM-6). Let $\hbar$ be a self map on $\mathfrak{X}$ satisfying

$$\mathcal{M}(\hbar\varkappa, \hbar y, k\mathfrak{t}) \geq \mathcal{M}(\varkappa, y, \mathfrak{t}), \tag{2.5.1}$$

for all $\varkappa$, y in $\mathfrak{X}$, $0 < k < 1$ and $\mathfrak{t} > 0$. Then, $\hbar$ has a unique fixed point in $\mathfrak{X}$.

Further, in [121], the monotonicity of $\mathcal{M}(\varkappa, y, \cdot)$ was also discussed in the form of following result:

Lemma 2.5.1 ([121]). Let (Ҳ, M, ∗) be a fuzzy metric space. Then, M(ϰ, y, ·) is non-decreasing for all ϰ, y ∈ Ҳ.

Later on, George and Veeramani [21] defined the topology induced by a fuzzy metric and redefined the definition of Cauchy sequence.

Definition 2.5.2 ([21]). A sequence $\{\varkappa_n\}$ in a fuzzy metric space (Ҳ, M, ∗) is a **Cauchy sequence** iff for each $\varepsilon > 0$ and ƫ > 0, there exists $n_0 \in \mathbb{N}$ such that

$$M(\varkappa_n, \varkappa_m, ƫ) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

Definition 2.5.3 ([21]). Let (Ҳ, M, ∗) be a fuzzy metric space. An **open ball** Ƀ(ϰ, ɍ, ƫ) with centre ϰ ∈ Ҳ and radius ɍ, 0 < ɍ < 1, ƫ > 0 is defined by

$$Ƀ(\varkappa, ɍ, ƫ) = \{y \in Ҳ : M(\varkappa, y, ƫ) > 1 - ɍ\}.$$

Further, George and Veeramani [21] also defined the topology τ on fuzzy metric space (Ҳ, M, ∗) by

$$\tau = \{A \subset Ҳ : \varkappa \in A \text{ iff there exist } ƫ > 0 \text{ and } ɍ \in (0, 1) \text{ such that } Ƀ(\varkappa, ɍ, ƫ) \subset A\}.$$

Theorem 2.5.2 ([21]). Every fuzzy metric space is Hausdorff.

L*ó*pez and Romaguera [129] proved the following lemma for the continuity of the function M on $Ҳ^2 \times \mathbb{R}^+ \backslash \{0\}$.

Lemma 2.5.2 ([129]). Let (Ҳ, M, ∗) be a fuzzy metric space. Then, M is a continuous function on $Ҳ^2 \times \mathbb{R}^+ \backslash \{0\}$.

Developing common fixed point results for various mappings including commuting, weakly commuting, R-weakly commuting, R-weakly commuting of type (A_f), (A_g) and (P), compatible, compatible of type (A), (B), (P), (C), (A_f), (A_g) and weakly compatible mappings has always been an area of great interest for researchers. Time-to-time, these notions have been extended from metric to fuzzy metric structure. Several results have been proved in this direction by various researchers (see, [130], [131], [132], [133] etc.) in fuzzy metric spaces.

In 2002, Aamri and El-Moutawakil [29] designed an important concept of property (E.A.) for pair of self mappings in metric spaces, which was later carried out in fuzzy metric spaces by Pant and Pant [134]. The significance of this property is that it affirms containment of ranges without the need of continuity of mappings and further, it minimizes the commutative assumption of the mappings to the commutative condition at their coincidence points. Moreover, it also allows the substitution of the completeness of the entire space with the closeness of the range subspace. Liu et al. [135] extended (E.A.) property to common property (E.A.) for a pair of single- and multi- valued maps in metric spaces. Later on, the common property (E.A.) was studied by Abbas et al. [136] in fuzzy metric spaces for pairs of self mappings.

In order to generalize the notion of property (E.A.), Sintunavarat and Kumam [30] introduced a new notion of "common limit in the range" property (or (CLR) property). The (CLR) property ensures that the necessity of the completeness of the space or range subspace can be relaxed entirely without the requirement of any other replacement. Chauhan et al. [137] extended the (CLR) property to "joint common limit in the range" property ((JCLR) property) of mappings and utilized it to formulate their results in fuzzy metric spaces. On the other hand, Chauhan [138] extended (CLR) property from single pair of self mappings to two pairs of self mappings and introduced "common limit in the range of mappings Ş and Ţ" property ($(CLR_{ŞŢ})$ property) in fuzzy metric spaces. Now-a-days, these notions are utilized rapidly for establishing fixed point results in the abstract spaces including fuzzy metric spaces (see [139], [140], [141], [142], [143], etc.).

In present times, fixed point theory is developing enormously in fuzzy metric spaces. After the innovation of the notion of coupled fixed points by Guo and Lakshmikantham [54], the problems concerning the computation of coupled fixed points were also given fuzzy treatment. Sedghi et al. [144] proved a coupled fixed point result under a contractive condition in fuzzy metric spaces. Later on, Zhu and Xiao [145] proved that the hypotheses considered by Sedghi et al. [144] to prove their result were incorrect. On the other hand, Hu [146] developed the fuzzy counterpart of the notion of compatible mappings for coupled fixed point problems and utilized the notion to obtain a common fixed point result under a ϕ – contraction in fuzzy metric spaces. Subsequently, coupled fixed point problems for ϕ – contractions in FM-spaces were discussed rapidly by various authors (see, [147], [63], etc.).

Now onwards, we use the term **FM-space** to denote fuzzy metric space.

Definition 2.5.4 ([146]). Let (Ҳ, Ϻ, ∗) be a FM-space. The mappings Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ are said to be **compatible** if

$$\lim_{n\to\infty} Ϻ(gҒ(\varkappa_n, y_n), Ғ(g\varkappa_n, gy_n), ƭ) = 1,$$

$$\lim_{n\to\infty} Ϻ(gҒ(y_n, \varkappa_n), Ғ(gy_n, g\varkappa_n), ƭ) = 1,$$

for all ƭ > 0, whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ҳ such that $\lim_{n\to\infty} Ғ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$ and $\lim_{n\to\infty} Ғ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y ∈ Ҳ.

In this case, we say that the pair (Ғ, g) is **compatible**.

Definition 2.5.5 ([146]). Denote by $\Phi_{\phi} = \{\phi: \mathbb{R}^+ \to \mathbb{R}^+\}$, the class of gauge functions, where each ϕ satisfies the followings:

(ϕ-1) ϕ is non-decreasing;

(ϕ-2) ϕ is upper semi-continuous from the right;

(ϕ-3) $\sum_{m=0}^{\infty} \phi^m(t) < \infty$ for all $t > 0$, where $\phi^{m+1}(t) = \phi(\phi^m(t))$, $m \in \mathbb{N}$.

Note that, "if $\phi \in \Phi_{\phi}$, then $\phi(t) < t$ for all $t > 0$".

Utilizing the gauge function $\phi \in \Phi_{\phi}$, Hu [146] established a common fixed point result in FM-spaces for the pair (F, g) of compatible mappings satisfying the following ϕ - contractive condition:

$$M(F(x, y), F(u, v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t), \tag{2.5.2}$$

for all x, y, u, v in X and $t > 0$.

Later on, Hu et al. [147] generalized the result of Hu [146] for a pair weakly compatible mappings, which was further generalized by Jain et al. [63] for two pairs ($A: X \times X \to X$, $S: X \to X$) and ($B: X \times X \to X$, $T: X \to X$) of weakly compatible mappings under the following condition:

$$M(A(x, y), B(u, v), \phi(t)) \geq M(Sx, Tu, t) * M(Sy, Tv, t), \tag{2.5.3}$$

for all x, y, u, v in X and $t > 0$, where $\phi \in \Phi_{\phi}$.

In the same paper [63], the authors have also introduced the notions of weakly commuting mappings and their variants including R-weakly commuting mappings, R-weakly commuting mappings of type (A_F), (A_g), (P) in context of coupled fixed point theory in FM-spaces. At the same time, Dalal and Masmali [148] studied the notions of variants of compatible mappings that includes compatible mappings of type (A), (B), (C), (P), (A_F), (A_g) in the context of coupled fixed point theory in FM-spaces and obtained some interesting results using these notions. All these notions and property (E.A.), common property (E.A.), (CLRg) property and (CLR_{ST}) property will be discussed later in the present work.

In modern times, researchers are continuous exploring new fundamentals in the theory of coupled fixed points in FM-spaces and the theory is growing rapidly in these spaces.

FRAMEWORK OF CHAPTER - III

In this chapter, we discuss coupled fixed point and coupled common fixed point results under (φ, ψ) – contractive conditions in POMS. Some coupled fixed point results in POPMS are also established. An application to the solution of an integral equation and a result of the integral type is also given.

PUBLISHED WORK:

(1) Journal of Inequalities and Applications, 2012, **2012**:285.

(2) International Journal of Analysis, vol. 2014, Article ID 586096, 9 pages, 2014.

(3) Journal of Mathematics and Computer Science, 7 (4) (2013), pp. 276-292.

CHAPTER – III

COUPLED FIXED POINTS FOR SYMMETRIC CONTRACTIVE CONDITIONS

This chapter deals with (φ, ψ) – contractive conditions in POMS and POPMS. The contractive conditions under consideration are symmetric in nature and weaken some of the already existing contractive conditions present in the literature. This chapter has five sections. Section 3.1 gives a brief introduction to symmetric contractive conditions. In section 3.2, we establish the existence and uniqueness of coupled common fixed points for mappings with MgMP satisfying a (φ, ψ) – contractive condition in POMS. Section 3.3 consists of coupled fixed point results under a (φ, ψ) – contractive condition in POMS. In section 3.4, we establish coupled fixed point result under symmetric (φ, ψ) – weakly contractive condition in the setup of POPMS. In the last section 3.5, an application to the existence and uniqueness of the solution of an integral equation is discussed. In this section, a result of the integral type is also given.

Author's Original Contributions In This Chapter Are:

Theorems: 3.2.1, 3.2.2, 3.3.1, 3.3.2, 3.3.3, 3.4.1, 3.4.2, 3.4.3, 3.5.1, 3.5.2.

Lemma: 3.2.1.

Definition: 3.4.1, 3.5.1.

Corollaries: 3.2.1, 3.2.2, 3.4.1.

Examples: 3.2.1, 3.2.2, 3.3.1, 3.4.1.

Remarks: 3.2.1, 3.2.2, 3.3.1, 3.4.1.

Assumption: 3.5.1.

3.1 INTRODUCTION

Recently, Berinde [149] obtained coupled fixed point results for the mixed monotone mapping Ғ: Ҳ × Ҳ → Ҳ subjected to a contractive condition which is

(i) symmetric in nature;

(ii) weaker than the contractive condition (2.1.14) due to Bhaskar and Lakshmikantham [55].

The main result established by Berinde [149] is as follows:

Theorem 3.1.1 ([149]). Let (Ӽ, ≼, ḓ) be a POCMS and Ғ: Ӽ × Ӽ → Ӽ be a mixed monotone mapping and there exists a ƙ ∈ [0, 1) such that for ϰ ≽ u, y ≼ v, we have

$$ḓ\big(Ғ(ϰ, y), Ғ(u, v)\big) + ḓ\big(Ғ(y, ϰ) + Ғ(v, u)\big) \le ƙ[ḓ(ϰ, u) + ḓ(y, v)]. \qquad (3.1.1)$$

Suppose that Ӽ has the following property:

(P3) "there exist two elements $ϰ_0$, y_0 ∈ Ӽ with either $ϰ_0 \preccurlyeq Ғ(ϰ_0, y_0)$ and $y_0 \succcurlyeq Ғ(y_0, ϰ_0)$, or $ϰ_0 \succcurlyeq Ғ(ϰ_0, y_0)$ and $y_0 \preccurlyeq Ғ(y_0, ϰ_0)$",

then, Ғ has a coupled fixed point in Ӽ.

In a subsequent paper, Berinde [150] extended the results of Bhaskar and Lakshmikantham [55] and Luong and Thuan [67], by weakening the involved contractive conditions.

Berinde [150] considered the following class of functions:

Definition 3.1.1 ([150]). Let *Φ* denote the class of functions *φ*: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(i_φ) *φ* is continuous and (strictly) increasing;

(ii_φ) *φ*(ƭ) < ƭ for all ƭ > 0;

(iii_φ) $\varphi(ƭ + s) \le \varphi(ƭ) + \varphi(s)$ for all ƭ, s ∈ $\mathbb{R}^+$.

If *φ* ∈ *Φ*, then *φ*(ƭ) = 0 iff ƭ = 0.

Berinde [150] also considered the class *Ψ* (originally, given by Luong and Thuan [67]) of functions *ψ*: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

(i_ψ) "$\lim_{ƭ \to ɽ} \psi(ƭ) > 0$ for all ɽ > 0 and $\lim_{ƭ \to 0+} \psi(ƭ) = 0$".

In order to prove his results, Berinde [150] utilized Assumption 2.1.7 which is again stated below (for convenience):

Assumption 2.1.7 ([55]). Ӽ has the property:

(i) "if a non-decreasing sequence $\{ϰ_n\}_{n=0}^{\infty} \subset$ Ӽ converges to ϰ, then $ϰ_n \preccurlyeq ϰ$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset$ Ӽ converges to y, then $y \preccurlyeq y_n$ for all n".

Berinde [150] proved the following coupled fixed point result which generalizes Theorems 2.1.14 and 2.1.19:

Theorem 3.1.2 ([150]). Let (Ӽ, ≼, ḓ) be a POCMS and Ғ: Ӽ × Ӽ → Ӽ be a mixed monotone mapping for which there exist *φ* ∈ *Φ* and *ψ* ∈ *Ψ* such that for all ϰ, y, u, v ∈ Ӽ with ϰ ≽ u, y ≼ v, we have

$$\varphi\left(\frac{ḓ\big(Ғ(ϰ,y), Ғ(u,v)\big) + ḓ\big(Ғ(y, ϰ), Ғ(v,u)\big)}{2}\right) \le \varphi\left(\frac{ḓ(ϰ,u) + ḓ(y,v)}{2}\right) - \psi\left(\frac{ḓ(ϰ,u) + ḓ(y,v)}{2}\right). \qquad (3.1.2)$$

Suppose either

(a) $Ꞙ$ is continuous, or (b) $Ҳ$ assumes Assumption 2.1.7.

If $Ҳ$ has the property (P3), then $Ꞙ$ has a coupled fixed point.

Berinde [150] also noted that, since the contractive condition (3.1.2) is valid only for comparable elements in $Ҳ^2$ (= $Ҳ \times Ҳ$), thus, in general, Theorem 3.1.2 cannot guarantee the uniqueness of the coupled fixed point. Therefore, it would be essential to associate some additional condition(s) to ensure the uniqueness of the coupled fixed point obtained in Theorem 3.1.2. Such kind of condition was used in [40].

Assumption 3.1.1 ([40]). "For all $Y = (\varkappa, y)$, $\widehat{Y} = (\hat{\varkappa}, \hat{y}) \in Ҳ^2$, there exists $\widehat{Z} = (z_1, z_2) \in Ҳ^2$ that is comparable to Y and $\widehat{Y}$".

Present chapter deals with the extension and generalization of the contractive conditions (3.1.1) and (3.1.2).We consider the non-empty set $Ҳ$ and the partial order $\preccurlyeq$ on $Ҳ$. Also, $Ꞙ: Ҳ \times Ҳ \to Ҳ$, $g: Ҳ \to Ҳ$ be two mappings. Now, let us recall some notations and definitions already given in the previous chapters that are useful in our work.

Property (P1): "There exist two elements $\varkappa_0, y_0 \in Ҳ$ with $\varkappa_0 \preccurlyeq Ꞙ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ꞙ(y_0, \varkappa_0)$".

Property (P2): "There exist two elements $\varkappa_0, y_0 \in Ҳ$ such that $g\varkappa_0 \preccurlyeq Ꞙ(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq Ꞙ(y_0, \varkappa_0)$".

Assumption 2.1.8 ([56]). $Ҳ$ has the property:

(i) "if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset Ҳ$ converges to $\varkappa$, then $g\varkappa_n \preccurlyeq g\varkappa$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset Ҳ$ converges to y, then $gy \preccurlyeq gy_n$ for all n".

3.2. COUPLED COMMON FIXED POINTS FOR (φ, ψ) - CONTRACTIVE CONDITION

In this section, we extend the results of Berinde [149, 150] (that is, Theorems 3.1.1 and 3.1.2) using a pair of compatible mappings that satisfies a $(\varphi, \psi) -$ contractive condition in POMS. The contractive condition under consideration weakens the contractive conditions involved in the results of Bhaskar and Lakshmikantham [55], Luong and Thuan [67] and Alotaibi and Alsulami [68] (that is, Theorems 2.1.14, 2.1.19 and 2.1.20, respectively).

We now present our main result as follows:

Theorem 3.2.1. Let (Ҳ, ≼, ḓ) be a POCMS and Ғ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be the mappings with Ғ having the MgMP on Ҳ. Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all ϰ, y, u, v ∈ Ҳ with gϰ ≽ gu and gy ≼ gv, we have

$$\varphi\left(\frac{ḓ(Ғ(ϰ,y), Ғ(u,v)) + ḓ(Ғ(y,ϰ), Ғ(v,u))}{2}\right)$$

$$\leq \varphi\left(\frac{ḓ(gϰ, gu) + ḓ(gy, gv)}{2}\right) - \psi\left(\frac{ḓ(gϰ, gu) + ḓ(gy, gv)}{2}\right). \quad (3.2.1)$$

Suppose that the mapping g is continuous, the pair (Ғ, g) is compatible and Ғ(Ҳ × Ҳ) ⊆ g(Ҳ). Assume either

(a) Ғ is continuous, or (b) Ҳ assumes Assumption 2.1.8.

If Ҳ has the property (P2), then Ғ and g have a coupled coincidence point in Ҳ.

Proof. Since Ҳ has the property (P2), so there exist $ϰ_0$, y_0 ∈ Ҳ such that $gϰ_0$ ≼ Ғ($ϰ_0$, y_0) and gy_0 ≽ Ғ(y_0, $ϰ_0$). As Ғ(Ҳ × Ҳ) ⊆ g(Ҳ), we choose $ϰ_1$, y_1 ∈ Ҳ such that $gϰ_1$ = Ғ($ϰ_0$, y_0), gy_1 = Ғ(y_0, $ϰ_0$). Similarly, we can choose $ϰ_2$, y_2 ∈ Ҳ such that $gϰ_2$ = Ғ($ϰ_1$, y_1), gy_2 = Ғ(y_1, $ϰ_1$).

Repeating this process, the sequences {$gϰ_n$} and {gy_n} can be obtained in Ҳ such that

$gϰ_{n+1}$ = Ғ($ϰ_n$, y_n), gy_{n+1} = Ғ(y_n, $ϰ_n$), for all n ≥ 0. (3.2.2)

Now, for all n ≥ 0, we show that

$gϰ_n$ ≼ $gϰ_{n+1}$, (3.2.3)

gy_n ≽ gy_{n+1}. (3.2.4)

As $gϰ_0$ ≼ Ғ($ϰ_0$, y_0) and gy_0 ≽ Ғ(y_0, $ϰ_0$), $gϰ_1$ = Ғ($ϰ_0$, y_0), gy_1 = Ғ(y_0, $ϰ_0$), we have $gϰ_0$ ≼ $gϰ_1$, gy_0 ≽ gy_1, so that (3.2.3) and (3.2.4) hold for n = 0.

Let (3.2.3) and (3.2.4) hold for some n > 0, that is, $gϰ_n$ ≼ $gϰ_{n+1}$, gy_n ≽ gy_{n+1}. Since Ғ satisfies the MgMP, by (3.2.2), we can get

$gϰ_{n+1}$ = Ғ($ϰ_n$, y_n) ≼ Ғ($ϰ_{n+1}$, y_n) ≼ Ғ($ϰ_{n+1}$, y_{n+1}) = $gϰ_{n+2}$,

gy_{n+1} = Ғ(y_n, $ϰ_n$) ≽ Ғ(y_{n+1}, $ϰ_n$) ≽ Ғ(y_{n+1}, $ϰ_{n+1}$) = gy_{n+2};

that is, $gϰ_{n+1}$ ≼ $gϰ_{n+2}$ and gy_{n+1} ≽ gy_{n+2}.

Now, by using mathematical induction, it follows that (3.2.3) and (3.2.4) hold for all n ≥ 0. If for some n ≥ 0, we have ($gϰ_{n+1}$, gy_{n+1}) = ($gϰ_n$, gy_n), then Ғ($ϰ_n$, y_n) = $gϰ_n$ and Ғ(y_n, $ϰ_n$) = gy_n, consequently, Ғ and g have a coupled coincidence point. So, we assume ($gϰ_{n+1}$, gy_{n+1}) ≠ ($gϰ_n$, gy_n), for all n ≥ 0, that is, we assume either $gϰ_{n+1}$ = Ғ($ϰ_n$, y_n) ≠ $gϰ_n$ or gy_{n+1} = Ғ(y_n, $ϰ_n$) ≠ gy_n.

As $gϰ_n$ ≽ $gϰ_{n-1}$ and gy_n ≼ gy_{n-1} for all n ≥ 1, by (3.2.1) and (3.2.2), we have

$$\varphi\left(\frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2}\right) = \varphi\left(\frac{d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))}{2}\right)$$

$$\leq \varphi\left(\frac{d(g x_n, g x_{n-1}) + d(g y_n, g y_{n-1})}{2}\right) - \psi\left(\frac{d(g x_n, g x_{n-1}) + d(g y_n, g y_{n-1})}{2}\right). \quad (3.2.5)$$

Since the function ψ is non-negative, by (3.2.5), we obtain that

$$\varphi\left(\frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2}\right) \leq \varphi\left(\frac{d(g x_n, g x_{n-1}) + d(g y_n, g y_{n-1})}{2}\right).$$

Then, using the monotone property of φ, we can obtain

$$\frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2} \leq \frac{d(g x_n, g x_{n-1}) + d(g y_n, g y_{n-1})}{2}.$$

Let $R_n = \frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2}$, then $\{R_n\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists some $R \geq 0$ such that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2}\right] = R. \quad (3.2.6)$$

Next, we claim $R = 0$. On the contrary, let us assume that $R > 0$. Taking $n \to \infty$ in (3.2.5) and using the properties of φ and ψ, we obtain

$$\varphi(R) = \lim_{n\to\infty} \varphi(R_n) \leq \lim_{n\to\infty} [\varphi(R_{n-1}) - \psi(R_{n-1})]$$

$$= \varphi(R) - \lim_{R_{n-1}\to R} \psi(R_{n-1}) < \varphi(R), \text{ a contradiction.}$$

Therefore, $R = 0$, so that, we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{d(g x_{n+1}, g x_n) + d(g y_{n+1}, g y_n)}{2}\right] = 0. \quad (3.2.7)$$

Now, we prove that $\{g x_n\}$ and $\{g y_n\}$ are Cauchy sequences.

If possible, let at least one of $\{g x_n\}$ and $\{g y_n\}$ is not a Cauchy sequence. So, there exits some $\varepsilon > 0$ for which we can find the sub-sequences $\{g x_{n(k)}\}$, $\{g x_{m(k)}\}$ of $\{g x_n\}$ and $\{g y_{n(k)}\}$, $\{g y_{m(k)}\}$ of $\{g y_n\}$ with $n(k) > m(k) \geq k$ such that

$$r_k = \frac{d(g x_{n(k)}, g x_{m(k)}) + d(g y_{n(k)}, g y_{m(k)})}{2} \geq \varepsilon. \quad (3.2.8)$$

Also, corresponding to $m(k)$, we can choose the smallest $n(k) \in \mathbb{N}$ with $n(k) > m(k) \geq k$ and satisfying (3.2.8). Then, we have

$$\frac{d(g x_{n(k)-1}, g x_{m(k)}) + d(g y_{n(k)-1}, g y_{m(k)})}{2} < \varepsilon. \quad (3.2.9)$$

Using (3.2.8), (3.2.9) and the triangle inequality, we obtain

$$\varepsilon \leq r_k = \frac{d(g x_{n(k)}, g x_{m(k)}) + d(g y_{n(k)}, g y_{m(k)})}{2}$$

$$\leq \frac{d(g x_{n(k)}, g x_{n(k)-1}) + d(g x_{n(k)-1}, g x_{m(k)}) + d(g y_{n(k)}, g y_{n(k)-1}) + d(g y_{n(k)-1}, g y_{m(k)})}{2}$$

$$< \frac{d(g x_{n(k)}, g x_{n(k)-1}) + d(g y_{n(k)}, g y_{n(k)-1})}{2} + \varepsilon.$$

Taking $k \to \infty$ and using (3.2.7) in the last inequality, we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\left[\frac{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right] = \varepsilon. \tag{3.2.10}$$

Again, using the triangle inequality, we have

$$r_k = \frac{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}$$

$$\leq \frac{\left\{\begin{array}{l} d(g\varkappa_{n(k)}, g\varkappa_{n(k)+1}) + d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}) + d(g\varkappa_{m(k)+1}, g\varkappa_{m(k)}) \\ + d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}) \end{array}\right\}}{2}$$

$$= R_{n(k)} + R_{m(k)} + \frac{d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}.$$

Now, using the monotone property of φ and the property (iii$_\varphi$), we get

$$\varphi(r_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \varphi\left(\frac{d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right). \tag{3.2.11}$$

Since $n(k) > m(k)$, $g\varkappa_{n(k)} \succcurlyeq g\varkappa_{m(k)}$ and $gy_{n(k)} \preccurlyeq gy_{m(k)}$, by (3.2.1) and (3.2.2), we have

$$\varphi\left(\frac{d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right)$$

$$= \varphi\left(\frac{d\big(F(\varkappa_{n(k)}, y_{n(k)}), F(\varkappa_{m(k)}, y_{m(k)})\big) + d\big(F(y_{n(k)}, \varkappa_{n(k)}), F(y_{m(k)}, \varkappa_{m(k)})\big)}{2}\right)$$

$$\leq \varphi\left(\frac{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right) - \psi\left(\frac{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right)$$

$$= \varphi(r_k) - \psi(r_k). \tag{3.2.12}$$

Using (3.2.11) and (3.2.12), we obtain

$$\varphi(r_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \varphi(r_k) - \psi(r_k).$$

Taking $k \to \infty$ in the last inequality, then using (3.2.7), (3.2.10) and the properties of φ and ψ, we obtain that

$$\varphi(\varepsilon) \leq \varphi(0) + \varphi(0) + \varphi(\varepsilon) - \lim_{k\to\infty}\psi(r_k)$$

$$= \varphi(\varepsilon) - \lim_{r_k \to \varepsilon}\psi(r_k) < \varphi(\varepsilon), \text{ a contradiction.}$$

Hence, both $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Now, by completeness of X, there exist some $\varkappa, y \in X$ such that

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa \text{ and } \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y. \tag{3.2.13}$$

Now, since the pair (F, g) is compatible, by (3.2.13), we obtain

$$\lim_{n\to\infty} d\big(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n)\big) = 0, \tag{3.2.14}$$

$$\lim_{n\to\infty} d\big(gF(y_n, \varkappa_n), F(gy_n, g\varkappa_n)\big) = 0. \tag{3.2.15}$$

Let us assume that assumption (a) holds.

Now, for all $n \geq 0$, we have

$$d(g\varkappa, F(g\varkappa_n, gy_n)) \leq d(g\varkappa, gF(\varkappa_n, y_n)) + d(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n)).$$

On taking $n \to \infty$ in the last inequality, then using (3.2.13) and (3.2.14) and the continuity of F and g, we can obtain $d(g\varkappa, F(\varkappa, y)) = 0$, so that we get $g\varkappa = F(\varkappa, y)$. Similarly, we can obtain $gy = F(y, \varkappa)$. Therefore, $(\varkappa, y)$ is a coupled coincidence point of F and g.

Next, assume that assumption (b) holds.

Using (3.2.3), (3.2.4) and (3.2.13), we obtain that $\{g\varkappa_n\}$ is a non-decreasing sequence converging to $\varkappa$ and $\{gy_n\}$ is a non-increasing sequence converging to y. Then, by assumption, for all $n \geq 0$, we get

$$gg\varkappa_n \preccurlyeq g\varkappa \text{ and } ggy_n \succcurlyeq gy. \tag{3.2.16}$$

Now, since the pair (F, g) is compatible and g is continuous, then using (3.2.13) and (3.2.15), we obtain

$$\lim_{n\to\infty} gg\varkappa_n = g\varkappa = \lim_{n\to\infty} gF(\varkappa_n, y_n) = \lim_{n\to\infty} F(g\varkappa_n, gy_n), \tag{3.2.17}$$

$$\lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} gF(y_n, \varkappa_n) = \lim_{n\to\infty} F(gy_n, g\varkappa_n). \tag{3.2.18}$$

Now, using the triangle inequality, we have

$$d(F(\varkappa, y), g\varkappa) \leq d(F(\varkappa, y), gg\varkappa_{n+1}) + d(gg\varkappa_{n+1}, g\varkappa),$$

or $$d(F(\varkappa, y), g\varkappa) \leq d(F(\varkappa, y), gF(\varkappa_n, y_n)) + d(gg\varkappa_{n+1}, g\varkappa).$$

On taking $n \to \infty$ in the last inequality and using (3.2.17), we can obtain

$$\begin{aligned} d(F(\varkappa, y), g\varkappa) &\leq \lim_{n\to\infty} d(F(\varkappa, y), gF(\varkappa_n, y_n)) + \lim_{n\to\infty} d(gg\varkappa_{n+1}, g\varkappa) \\ &\leq \lim_{n\to\infty} d(F(\varkappa, y), F(g\varkappa_n, gy_n)). \end{aligned} \tag{3.2.19}$$

Similarly, we get

$$d(F(y, \varkappa), gy) \leq \lim_{n\to\infty} d(F(y, \varkappa), F(gy_n, g\varkappa_n)). \tag{3.2.20}$$

Using (3.2.19), (3.2.20) and the property (i_φ), we get

$$\varphi\left(\frac{d(F(\varkappa,y), g\varkappa) + d(F(y,\varkappa), gy)}{2}\right) \leq \lim_{n\to\infty} \varphi\left(\frac{d(F(\varkappa,y), F(g\varkappa_n, gy_n)) + d(F(y,\varkappa), F(gy_n, g\varkappa_n))}{2}\right). \tag{3.2.21}$$

Using (3.2.1) and (3.2.16), we obtain that

$$\begin{aligned} &\varphi\left(\frac{d(F(\varkappa,y), F(g\varkappa_n, gy_n)) + d(F(y,\varkappa), F(gy_n, g\varkappa_n))}{2}\right) \\ &\leq \varphi\left(\frac{d(g\varkappa, gg\varkappa_n) + d(gy, ggy_n)}{2}\right) - \psi\left(\frac{d(g\varkappa, gg\varkappa_n) + d(gy, ggy_n)}{2}\right). \end{aligned} \tag{3.2.22}$$

Using (3.2.22) in (3.2.21), we get

$$\varphi\left(\frac{ḓ(Ғ(\varkappa,y), g\varkappa) + ḓ(Ғ(y,\varkappa), gy)}{2}\right)$$

$$\leq \lim_{n\to\infty}\varphi\left(\frac{ḓ(g\varkappa, gg\varkappa_n) + ḓ(gy, ggy_n)}{2}\right) - \lim_{n\to\infty}\psi\left(\frac{ḓ(g\varkappa, gg\varkappa_n) + ḓ(gy, ggy_n)}{2}\right).$$

Using (3.2.17), (3.2.18), the continuity of φ and $\lim_{t\to 0+}\psi(t) = 0$, we obtain

$$\varphi\left(\frac{ḓ(Ғ(\varkappa,y), g\varkappa) + ḓ(Ғ(y,\varkappa), gy)}{2}\right) \leq \lim_{n\to\infty}\varphi\left(\frac{ḓ(g\varkappa, gg\varkappa_n) + ḓ(gy, ggy_n)}{2}\right) = \varphi(0) = 0.$$

Since φ is a non-negative function with $\varphi(0) = 0$, so by last inequality we can get ḓ(Ғ($\varkappa$, y), g$\varkappa$) = 0 and ḓ(Ғ(y, $\varkappa$), gy) = 0, so that, we have Ғ($\varkappa$, y) = g$\varkappa$ and Ғ(y, $\varkappa$) = gy. Therefore, ($\varkappa$, y) is a coupled coincidence point of Ғ and g.

Remark 3.2.1. Theorem 3.2.1 extends Theorem 3.1.2 (Berinde [150]). Considering g to be the identity mapping in Theorem 3.2.1, we can obtain Theorem 3.1.2.

The following example furnishes that the contractive condition (3.2.1) of Theorem 3.2.1 weakens condition (2.1.19) of Theorem 2.1.20, which implies that Theorem 3.2.1 is more general than Theorem 2.1.20 (Alotaibi and Alsulami [68]).

Example 3.2.1. Let Ӽ = ℝ, then, (Ӽ, ≼, ḓ) is a POCMS, with partial ordering ≼ being the usual ordering ≤ of real numbers and ḓ: Ӽ × Ӽ → $\mathbb{R}^+$ defined by ḓ($\varkappa$, y) = $|\varkappa - y|$ for $\varkappa$, y ∈ Ӽ. Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be defined by Ғ($\varkappa$, y) = $\frac{\varkappa-5y}{20}$ for $\varkappa$, y ∈ Ӽ and g$\varkappa$ = $\frac{\varkappa}{2}$ for $\varkappa$ ∈ Ӽ, respectively. Then, Ғ is continuous and has MgMP. Also, Ғ(Ӽ × Ӽ) ⊆ g(Ӽ) and the pair (Ғ, g) is compatible. Further, Ғ and g satisfy the condition (3.2.1) but does not satisfy (2.1.19). On the contrary, assume that there exist some $\varphi \in \Phi_1$ and $\psi \in \Psi$ such that (2.1.19) holds. Then, for $\varkappa$, y, u, v in Ӽ with g$\varkappa$ ≥ gu and gy ≤ gv, we have

$$\varphi\left(ḓ\big(Ғ(\varkappa,y),Ғ(u,v)\big)\right) \leq \frac{1}{2}\varphi\big(ḓ(g\varkappa, gu) + ḓ(gy, gv)\big) - \psi\left(\frac{ḓ(g\varkappa, gu) + ḓ(gy, gv)}{2}\right),$$

that is, $$\varphi\left(\left|\frac{\varkappa-5y}{20} - \frac{u-5v}{20}\right|\right) \leq \frac{1}{2}\varphi\left(\left|\frac{\varkappa}{2} - \frac{u}{2}\right| + \left|\frac{y}{2} - \frac{v}{2}\right|\right) - \psi\left(\frac{\left|\frac{\varkappa}{2} - \frac{u}{2}\right| + \left|\frac{y}{2} - \frac{v}{2}\right|}{2}\right)$$

$$= \frac{1}{2}\varphi\left(\frac{|\varkappa-u|+|y-v|}{2}\right) - \psi\left(\frac{|\varkappa-u|+|y-v|}{4}\right).$$

Taking $\varkappa$ = u, y ≠ v and $\varrho = \frac{|y-v|}{4}$ in the last inequality, we get $\varphi(\varrho) \leq \frac{1}{2}\varphi(2\varrho) - \psi(\varrho)$, $\varrho > 0$. Using (φ_3), we obtain $\frac{1}{2}\varphi(2\varrho) \leq \varphi(\varrho)$ and hence, for all $\varrho > 0$, we deduce that $\psi(\varrho) \leq 0$, so that we have $\psi(\varrho) = 0$, a contradiction to (i_ψ). Therefore, Ғ and g do not satisfy (2.1.19), so that Theorem 2.1.20 does not hold here.

We now show that (3.2.1) holds. For, $\varkappa$ ≥ u and y ≤ v, we have

$$\left|\frac{\varkappa-5y}{20} - \frac{u-5v}{20}\right| \leq \frac{1}{20}|\varkappa - u| + \frac{1}{4}|y - v| \text{ and } \left|\frac{y-5\varkappa}{20} - \frac{v-5u}{20}\right| \leq \frac{1}{20}|y - v| + \frac{1}{4}|\varkappa - u|.$$

Adding the last two inequalities, we can exactly obtain (3.2.1) for $\varphi(\mathfrak{t}) = \frac{1}{2}\mathfrak{t}$ and $\psi(\mathfrak{t}) = \frac{1}{5}\mathfrak{t}$. Further, $\varkappa_0$ (= −1), y_0 (= 1) ∈ Ҳ satisfying property (P2). Now, the mappings Ғ, g, φ and ψ meet the requirements of Theorem 3.2.1. By applying Theorem 3.2.1, the mappings Ғ and g have a coupled coincidence point (0, 0) in Ҳ. But Theorem 2.1.20 cannot be applied to Ғ and g in this example.

Corollary 3.2.1. Let (Ҳ, ≼, ḑ) be a POCMS and Ғ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be the mappings with Ғ having the MgMP on Ҳ. Let there exists some ķ, 0 ≤ ķ < 1 such that for all ϰ, y, u, v in Ҳ with gϰ ≽ gu and gy ≼ gv, we have

$$ḑ\big(Ғ(\varkappa, y), Ғ(u,v)\big) + ḑ\big(Ғ(y, \varkappa), Ғ(v, u)\big) \leq ķ[ḑ(g\varkappa, gu) + ḑ(gy, gv)]. \quad (3.2.23)$$

Assume that g is continuous, the pair (Ғ, g) is compatible and Ғ(Ҳ × Ҳ) ⊆ g(Ҳ). Assume either

(a) Ғ is continuous, or (b) Ҳ assumes Assumption 2.1.8.

If Ҳ has the property (P2), then Ғ and g have a coupled coincidence point in Ҳ.

Proof. Considering $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$ and $\psi(\mathfrak{t}) = (1 - ķ)\frac{\mathfrak{t}}{2}$, 0 ≤ ķ < 1 in Theorem 3.2.1, we can obtain the required result.

Remark 3.2.2. (i) Corollary 3.2.1 extends Theorem 3.1.1 (Berinde [149]) for a pair of compatible mappings.

(ii) Example 3.2.1 also supports Corollary 3.2.1 for ķ = $\frac{3}{5}$. Consequently, Corollary 3.2.1 is more general than Theorem 2.1.20 (Alotaibi and Alsulami [68]).

Corollary 3.2.2. Let (Ҳ, ≼, ḑ) be a POCMS and Ғ: Ҳ × Ҳ → Ҳ be a mapping with MMP on Ҳ. Let there exists some ķ, 0 ≤ ķ < 1 such that for all ϰ, y, u, v in Ҳ with ϰ ≽ u and y ≼ v, we have

$$ḑ\big(Ғ(\varkappa, y), Ғ(u,v)\big) + ḑ\big(Ғ(y, \varkappa), Ғ(v, u)\big) \leq ķ[ḑ(\varkappa, u) + ḑ(y, v)]. \quad (3.2.24)$$

Assume either

(a) Ғ is continuous, or (b) Ҳ assumes Assumption 2.1.7.

If Ҳ has the property (P1), then Ғ has a coupled fixed point in Ҳ.

Proof. Considering g to be the identity mapping on Ҳ in Corollary 3.2.1, we can obtain the required result.

The following example furnishes that the contractive condition (3.2.24) of Corollary 3.2.2 weakens conditions (2.1.14) of Theorem 2.1.14 and (2.1.18) of Theorem 2.1.19, so that Corollary 3.2.2 is more general than Theorem 2.1.14

(Bhaskar and Lakshmikantham [55]) and Theorem 2.1.19 (Luong and Thuan [67]), respectively.

Example 3.2.2. Let $\mathfrak{X} = \mathbb{R}$, then, $(\mathfrak{X}, \preccurlyeq, d)$ is a POCMS, with partial ordering $\preccurlyeq$ being the usual ordering $\leq$ of real numbers and $d: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ defined by $d(\varkappa, y) = |\varkappa - y|$ for $\varkappa, y \in \mathfrak{X}$. Let $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be defined by $F(\varkappa, y) = \frac{\varkappa - 3y}{6}$ for $\varkappa, y \in \mathfrak{X}$. Then, F is continuous, has MMP and satisfies the condition (3.2.24) but does not satisfy any of the conditions (2.1.14) and (2.1.18), so that Theorems 2.1.14 and 2.1.19 do not hold here. Let there exists some $k \in [0, 1)$ such that (2.1.14) holds, so that for $\varkappa \geq u$ and $y \leq v$, we shall have

$$d(F(\varkappa, y), F(u, v)) \leq \frac{k}{2}\,[d(\varkappa, u) + d(y, v)],$$

that is,
$$\left|\frac{\varkappa - 3y}{6} - \frac{u - 3v}{6}\right| \leq \frac{k}{2}\{|\varkappa - u| + |y - v|\},$$

from which, for $\varkappa = u$, we can obtain $|y - v| \leq k\,|y - v|$, $y \leq v$, which for $y < v$ implies that $1 \leq k$, a contradiction, since $k \in [0, 1)$. Therefore, F does not satisfy (2.1.14).

Also, the condition (2.1.18) is not satisfied. On the contrary, assume that there exist some $\varphi \in \Phi_1$ and $\psi \in \Psi$ such that (2.1.18) holds. Then, for all $\varkappa \geq u$ and $y \leq v$, we shall have

$$\varphi\left(d\big(F(\varkappa, y), F(u, v)\big)\right) \leq \frac{1}{2}\,\varphi\big(d(\varkappa, u) + d(y, v)\big) - \psi\left(\frac{d(\varkappa, u) + d(y, v)}{2}\right),$$

$$\varphi\left(\left|\frac{\varkappa - 3y}{6} - \frac{u - 3v}{6}\right|\right) \leq \frac{1}{2}\,\varphi(|\varkappa - u| + |y - v|) - \psi\left(\frac{|\varkappa - u| + |y - v|}{2}\right).$$

Taking $\varkappa = u$, $y \neq v$ and $\varrho = \frac{|y - v|}{2}$ in the last inequality, we get

$$\varphi(\varrho) \leq \frac{1}{2}\,\varphi(2\varrho) - \psi(\varrho),\ \varrho > 0.$$

Using (φ_3), we can obtain $\frac{1}{2}\,\varphi(2\varrho) \leq \varphi(\varrho)$. Therefore, for all $\varrho > 0$, we can deduce that $\psi(\varrho) \leq 0$, so that, we get $\psi(\varrho) = 0$, a contradiction to (i_ψ). Hence, F does not satisfy (2.1.18). Next, we shall prove that (3.2.24) holds. For, $\varkappa \geq u$ and $y \leq v$, we have

$$\left|\frac{\varkappa - 3y}{6} - \frac{u - 3v}{6}\right| \leq \frac{1}{6}\,|\varkappa - u| + \frac{1}{2}\,|y - v| \ \text{ and } \ \left|\frac{y - 3\varkappa}{6} - \frac{v - 3u}{6}\right| \leq \frac{1}{6}\,|y - v| + \frac{1}{2}\,|\varkappa - u|.$$

Adding the last two inequalities, we can exactly obtain (3.2.24) for $k = \frac{2}{3}$.

Further, $\varkappa_0$ $(= -1)$, y_0 $(= 1)$ $\in \mathfrak{X}$ such that the property (P1) holds. By applying Corollary 3.2.2, we can obtain that F has a coupled fixed point $(0, 0)$ in $\mathfrak{X}$. But the Theorems 2.1.14 and 2.1.19 cannot be applied to F in this example.

Coupled Common Fixed Points

Next, we shall obtain the existence and uniqueness of the coupled common fixed point under the hypotheses of Theorem 3.2.1. For, we require the followings:

Assumption 3.2.1 ([59]). "For every $(\varkappa, y)$, $(\varkappa^*, y^*) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(\varkappa, y), F(y, x))$ and $(F(\varkappa^*, y^*), F(y^*, \varkappa^*))$".

Lemma 3.2.1. Let $F: X \times X \to X$ and $g: X \to X$ be the mappings such that the pair (F, g) is compatible. If there exists some $(\varkappa, y) \in X \times X$ such that $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$, then $gF(\varkappa, y) = F(g\varkappa, gy)$ and $gF(y, \varkappa) = F(gy, g\varkappa)$.

Or in simple words, "The pair of compatible mappings $F: X \times X \to X$ and $g: X \to X$ commutes at their coincidence points".

Proof. Since the pair (F, g) is compatible, we have

$$\lim_{n\to\infty} d\big(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n)\big) = 0,$$

$$\lim_{n\to\infty} d\big(gF(y_n, \varkappa_n), F(gy_n, g\varkappa_n)\big) = 0,$$

whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = a$ and $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = b$ for some a, b in X.

Considering $\varkappa_n = \varkappa$, $y_n = y$ and using $g\varkappa = F(\varkappa, y)$, $gy = F(y, \varkappa)$, it follows that

$$d(gF(\varkappa, y), F(g\varkappa, gy)) = 0 \text{ and } d(gF(y, \varkappa), F(gy, g\varkappa)) = 0.$$

Therefore, $gF(\varkappa, y) = F(g\varkappa, gy)$ and $gF(y, \varkappa) = F(gy, g\varkappa)$.

Theorem 3.2.2. In addition to the hypotheses of Theorem 3.2.1, suppose that the Assumption 3.2.1 also holds. Then, F and g have unique coupled common fixed point in X.

Proof. By Theorem 3.2.1, the set of coupled coincidence points is non-empty. Now, to prove the result, we first show that if $(\varkappa, y)$ and $(\varkappa^*, y^*)$ are coupled coincidence points, then

$$g\varkappa = g\varkappa^* \text{ and } gy = gy^*. \tag{3.2.25}$$

By Assumption 3.2.1, there exists some $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(\varkappa, y), F(y, \varkappa))$ and $(F(\varkappa^*, y^*), F(y^*, \varkappa^*))$. Take $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$.

Now, as in the proof of Theorem 3.2.1, inductively, the sequences $\{gu_n\}$ and $\{gv_n\}$ can be defined such that $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$.

Taking $\varkappa_0 = \varkappa$, $y_0 = y$, $\varkappa_0^* = \varkappa^*$, $y_0^* = y^*$, then, on the same way, we can define the sequences $\{g\varkappa_n\}$, $\{gy_n\}$ and $\{g\varkappa_n^*\}$, $\{gy_n^*\}$. Now, it can be easily shown that $g\varkappa_{n+1} = F(\varkappa_n, y_n)$, $gy_{n+1} = F(y_n, \varkappa_n)$ and $g\varkappa_{n+1}^* = F(\varkappa_n^*, y_n^*)$, $gy_{n+1}^* = F(y_n^*, \varkappa_n^*)$ for all $n \geq 0$. Also, since $(F(u, v), F(v, u)) = (gu_1, gv_1)$ and $(F(\varkappa, y), F(y, \varkappa)) = (g\varkappa_1, gy_1) = (g\varkappa, gy)$ are comparable, we have $gu_1 \succcurlyeq g\varkappa$ and $gv_1 \preccurlyeq gy$. Now, it is easy to obtain that (gu_n, gv_n) and $(g\varkappa, gy)$ are comparable, so that $gu_n \succcurlyeq g\varkappa$ and $gv_n \preccurlyeq gy$ for all $n \geq 1$. Then, by (3.2.1), we obtain

$$\varphi\left(\frac{d(gu_{n+1}, g\varkappa) + d(gv_{n+1}, gy)}{2}\right) = \varphi\left(\frac{d(F(u_n,v_n), F(\varkappa,y)) + d(F(v_n,u_n), F(y,\varkappa))}{2}\right)$$

$$\leq \varphi\left(\frac{d(gu_n, g\varkappa) + d(gv_n, gy)}{2}\right) - \psi\left(\frac{d(gu_n, g\varkappa) + d(gv_n, gy)}{2}\right). \qquad (3.2.26)$$

Now, since ψ is a non-negative function, we get

$$\varphi\left(\frac{d(gu_{n+1}, g\varkappa) + d(gv_{n+1}, gy)}{2}\right) \leq \varphi\left(\frac{d(gu_n, g\varkappa) + d(gv_n, gy)}{2}\right).$$

Using the monotone property of φ, we can obtain

$$\frac{d(gu_{n+1}, g\varkappa) + d(gv_{n+1}, gy)}{2} \leq \frac{d(gu_n, g\varkappa) + d(gv_n, gy)}{2}. \qquad (3.2.27)$$

Let $d_n = \frac{d(gu_n, g\varkappa) + d(gv_n, gy)}{2}$. Then, $\{d_n\}$ is a monotonically decreasing sequence of non-negative real numbers. Therefore, there exists some $d \geq 0$ such that $\lim_{n\to\infty} d_n = d$.

We claim that $d = 0$. On the contrary, let us assume that $d > 0$. Now, on letting $n \to \infty$, in (3.2.26) and using the continuity of φ, we get

$$\varphi(d) \leq \varphi(d) - \lim_{d_n \to d} \psi(d_n) < \varphi(d), \text{ a contradiction.}$$

Therefore, $d = 0$, so that $\lim_{n\to\infty} d_n = 0$. Consequently, we get $gu_n \to g\varkappa$, $gv_n \to gy$ as $n \to \infty$. Similarly, we can obtain that $gu_n \to g\varkappa^*$, $gv_n \to gy^*$ as $n \to \infty$. Now, by uniqueness of limit, we can get $g\varkappa = g\varkappa^*$ and $gy = gy^*$. Hence, we have proved (3.2.25).

Also, since $g\varkappa = F(\varkappa, y)$, $gy = F(y, \varkappa)$ and the pair (F, g) is compatible, then using the Lemma 3.2.1, we get

$$gg\varkappa = gF(\varkappa, y) = F(g\varkappa, gy) \text{ and } ggy = gF(y, \varkappa) = F(gy, g\varkappa). \qquad (3.2.28)$$

Let us denote by $g\varkappa = z$ and $gy = w$. Then, using (3.2.28), we can obtain

$$gz = F(z, w) \text{ and } gw = F(w, z). \qquad (3.2.29)$$

Therefore, (z, w) is a coupled coincidence point of F and g. Now, using (3.2.25) with $\varkappa^* = z$ and $y^* = w$, we can obtain that $gz = g\varkappa$ and $gw = gy$, so that

$$gz = z \text{ and } gw = w. \qquad (3.2.30)$$

Using (3.2.29) and (3.2.30), we get $z = gz = F(z, w)$ and $w = gw = F(w, z)$.

Therefore, (ʐ, w) is the coupled common fixed point of Ғ and g. For uniqueness, let (e, l) be any coupled common fixed point of Ғ and g. Then, using (3.2.25), we can obtain e = ge = gʐ = ʐ and l = gl = gw = w. Then by (3.2.25), we have e = ge = gʐ = ʐ and l = gl = gw = w. Hence, the result is proved.

3.3 COUPLED FIXED POINTS FOR (φ, ψ) – CONTRACTIVE CONDITION

In this section, by considering a new (φ, ψ) – contractive condition in POMS, we generalize the results of Berinde [149, 150] (that is, Theorems 3.1.1 and 3.1.2, respectively) and weaken the contractive conditions involved in the results of Bhaskar and Lakshmikantham [55], Luong and Thuan [67] (that is, Theorems 2.1.14 and 2.1.19, respectively).

Before giving our results, we shall consider the following notions:

Let Φ_3 denote the class of all functions φ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(φ_i) φ is lower semi-continuous and (strictly) increasing;

(φ_{ii}) φ (ƭ) < ƭ for all ƭ > 0;

(φ_{iii}) φ(ƭ + s) ≤ φ(ƭ) + φ(s) for all ƭ, s ∈ $\mathbb{R}^+$.

Note that "$\lim_{n\to\infty} \varphi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$ for $t_n \in \mathbb{R}^+$".

Also, for $\varphi \in \Phi_3$, let Ψ_φ denote the class of all functions ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_i) $\limsup_{n\to\infty} \psi(t_n) < \varphi(r)$ if $\lim_{n\to\infty} t_n = r > 0$;

(ψ_{ii}) $\lim_{n\to\infty} \psi(t_n) = 0$ if $\lim_{n\to\infty} t_n = 0$ for $t_n \in \mathbb{R}^+$.

Now, we shall prove our results.

Theorem 3.3.1. Let (Ӽ, ≼, ḑ) be a POCMS and Ғ: Ӽ × Ӽ → Ӽ be a mapping with MMP on Ӽ and there exist $\varphi \in \Phi_3$ and $\psi \in \Psi_\varphi$ such that for all ϰ, y, u, v ∈ Ӽ with ϰ ≽ u and y ≼ v (or ϰ ≼ u and y ≽ v), we have

$$\varphi\left(\frac{ḑ(Ғ(ϰ,y), Ғ(u,v)) + ḑ(Ғ(y,ϰ), Ғ(v,u))}{2}\right) \le \psi\left(\frac{ḑ(ϰ,u) + ḑ(y,v)}{2}\right). \quad (3.3.1)$$

Suppose either

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ӽ has the property (P3), then Ғ has a coupled fixed point in Ӽ.

Proof. Since Ӽ has the property (P3), W.L.O.G. let there exist $ϰ_0$, y_0 ∈ Ӽ such that $ϰ_0$ ≼ Ғ($ϰ_0$, y_0) and y_0 ≽ Ғ(y_0, $ϰ_0$). Put $ϰ_1$ = Ғ($ϰ_0$, y_0) and y_1 = Ғ(y_0, $ϰ_0$). Then, we have

$\varkappa_0 \preccurlyeq \varkappa_1$ and $y_0 \succcurlyeq y_1$. Similarly, put $\varkappa_2 = F(\varkappa_1, y_1)$ and $y_2 = F(y_1, \varkappa_1)$. Since F has MMP, we get $\varkappa_1 \preccurlyeq \varkappa_2$ and $y_1 \succcurlyeq y_2$. Repeating this process, we can construct two sequences $\{\varkappa_n\}$ and $\{y_n\}$ in X such that $\varkappa_{n+1} = F(\varkappa_n, y_n)$ and $y_{n+1} = F(y_n, \varkappa_n)$ with

$$\varkappa_n \preccurlyeq \varkappa_{n+1}, \quad y_n \succcurlyeq y_{n+1}, \text{ for all } n \geq 0. \tag{3.3.2}$$

If $(\varkappa_{n+1}, y_{n+1}) = (\varkappa_n, y_n)$ for some $n \geq 0$, then we get $F(\varkappa_n, y_n) = \varkappa_n$ and $F(y_n, \varkappa_n) = y_n$, so that F has a coupled fixed point. So, we assume that $(\varkappa_{n+1}, y_{n+1}) \neq (\varkappa_n, y_n)$, for all $n \geq 0$, that is, we assume either $\varkappa_{n+1} = F(\varkappa_n, y_n) \neq \varkappa_n$ or $y_{n+1} = F(y_n, \varkappa_n) \neq y_n$.

Since $\varkappa_n \preccurlyeq \varkappa_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for $n \geq 0$, on applying (3.3.1), we get

$$\varphi\left(\frac{d(\varkappa_{n+1}, \varkappa_{n+2}) + d(y_{n+1}, y_{n+2})}{2}\right) = \varphi\left(\frac{d\big(F(\varkappa_n, y_n), F(\varkappa_{n+1}, y_{n+1})\big) + d\big(F(y_n, \varkappa_n), F(y_{n+1}, \varkappa_{n+1})\big)}{2}\right)$$

$$\leq \psi\left(\frac{d(\varkappa_n, \varkappa_{n+1}) + d(y_n, y_{n+1})}{2}\right), \tag{3.3.3}$$

Then, for all $n \geq 0$, we get

$$\varphi(R_{n+1}) \leq \psi(R_n), \tag{3.3.4}$$

where $R_n = \frac{d(\varkappa_n, \varkappa_{n+1}) + d(y_n, y_{n+1})}{2}$.

Also $R_n > 0$ for all $n \geq 0$. By (3.3.4), for any $n \geq 0$, we have

$$\varphi(R_{n+1}) \leq \psi(R_n) < \varphi(R_n). \tag{3.3.5}$$

Then, using the monotone property of φ, from (3.3.5), we can obtain that $\{R_n\}$ is a decreasing sequence of non-negative real numbers. So, there exists some $R \geq 0$, such that $\lim_{n\to\infty} R_n = R$. If $R > 0$, then by the properties of φ and ψ, we obtain

$$\varphi(R) \leq \limsup_{n\to\infty} \varphi(R_{n+1}) \leq \limsup_{n\to\infty} \psi(R_n) < \varphi(R),$$

a contradiction. Therefore $R = 0$ and hence, we get

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{d(\varkappa_n, \varkappa_{n+1}) + d(y_n, y_{n+1})}{2} = 0. \tag{3.3.6}$$

We now claim that $\{\varkappa_n\}$ and $\{y_n\}$ are Cauchy sequences. On the contrary, assume at least one of $\{\varkappa_n\}$, $\{y_n\}$ is not a Cauchy sequence. So, there exists some $\varepsilon > 0$ for which we can find sub-sequences $\{\varkappa_{n(k)}\}$, $\{\varkappa_{m(k)}\}$ of $\{\varkappa_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$r_k = \frac{d\big(\varkappa_{n(k)}, \varkappa_{m(k)}\big) + d\big(y_{n(k)}, y_{m(k)}\big)}{2} \geq \varepsilon. \tag{3.3.7}$$

Also, corresponding to $m(k)$, we smallest $n(k) \in \mathbb{N}$ with $n(k) > m(k) \geq k$ and satisfying (3.3.7). Then, we have

$$\frac{d\big(\varkappa_{n(k)-1}, \varkappa_{m(k)}\big) + d\big(y_{n(k)-1}, y_{m(k)}\big)}{2} < \varepsilon. \tag{3.3.8}$$

By (3.3.7), (3.3.8) and the triangle inequality, we get

$$\varepsilon \leq \mathfrak{r}_k = \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}$$

$$\leq \frac{d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)})}{2}$$

$$< \frac{d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})}{2} + \varepsilon.$$

On taking $k \to \infty$ and using (3.3.6) in the last inequality, we get

$$\lim_{k\to\infty} \mathfrak{r}_k = \lim_{k\to\infty}\left[\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right] = \varepsilon. \quad (3.3.9)$$

Now, using the triangle inequality, we have

$$\mathfrak{r}_k = \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}$$

$$\leq \frac{\left\{\begin{array}{l} d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\ + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \end{array}\right\}}{2}$$

$$= R_{n(k)} + R_{m(k)} + \frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})}{2}. \quad (3.3.10)$$

By monotone property of φ and the property (φ_{iii}), we get

$$\varphi(\mathfrak{r}_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \varphi\left(\frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})}{2}\right). \quad (3.3.11)$$

Since n(k) > m(k), we have $x_{n(k)} \succcurlyeq x_{m(k)}$ and $y_{n(k)} \preccurlyeq y_{m(k)}$.

Then, using (3.3.1), we get

$$\varphi\left(\frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})}{2}\right)$$

$$= \varphi\left(\frac{d\big(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})\big) + d\big(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)})\big)}{2}\right)$$

$$\leq \psi\left(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right)$$

$$= \psi(\mathfrak{r}_k). \quad (3.3.12)$$

Now, by (3.3.11) and (3.3.12), we get

$$\varphi(\mathfrak{r}_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \psi(\mathfrak{r}_k).$$

Since the function φ is lower semi-continuous, then letting $k \to \infty$ in the last inequality, we obtain that

$$\varphi(\varepsilon) \leq \lim_{k\to\infty} \sup \varphi(\mathfrak{r}_k)$$

$$\leq \lim_{k\to\infty} \varphi(R_{n(k)}) + \lim_{k\to\infty} \varphi(R_{n(k)}) + \lim_{k\to\infty} \sup \psi(\mathfrak{r}_k) < \varphi(\varepsilon),$$

a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Now, by completeness of X, there exist some $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Let us assume that assumption (a) holds.

Then, $\varkappa = \lim_{n\to\infty} \varkappa_{n+1} = \lim_{n\to\infty} F(\varkappa_n, y_n) = F(\varkappa, y)$,

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, \varkappa_n) = F(y, \varkappa),$$

which implies that $(\varkappa, y)$ is a coupled fixed point of F.

Next, assume that assumption (b) holds.

As the sequence $\{\varkappa_n\}$ is non-decreasing and convergent to $\varkappa$, by assumption, we get $\varkappa_n \preccurlyeq \varkappa$ for all n. Similarly, we have $y_n \succcurlyeq y$ for all n.

Then, we have

$$d(\varkappa, F(\varkappa, y)) \leq d(\varkappa, \varkappa_{n+1}) + d(\varkappa_{n+1}, F(\varkappa, y))$$
$$= d(\varkappa, \varkappa_{n+1}) + d(F(\varkappa_n, y_n), F(\varkappa, y))$$

and

$$d(y, F(y, \varkappa)) \leq d(y, y_{n+1}) + d(y_{n+1}, F(y, \varkappa))$$
$$= d(y, y_{n+1}) + d(F(y_n, \varkappa_n), F(y, \varkappa)).$$

So, we get

$$d(\varkappa, F(\varkappa, y)) - d(\varkappa, \varkappa_{n+1}) \leq d(F(\varkappa_n, y_n), F(\varkappa, y))$$

and $d(y, F(y, \varkappa)) - d(y, y_{n+1}) \leq d(F(y_n, \varkappa_n), F(y, \varkappa))$,

therefore, $\frac{1}{2}\left[d(\varkappa, F(\varkappa, y)) - d(\varkappa, \varkappa_{n+1}) + d(y, F(y, \varkappa)) - d(y, y_{n+1})\right]$

$$\leq \frac{1}{2}\left[d(F(\varkappa_n, y_n), F(\varkappa, y)) + d(F(y_n, \varkappa_n), F(y, \varkappa))\right],$$

which implies, by using the monotone property of φ and (3.3.1), that

$$\varphi\left(\frac{1}{2}\left[d(\varkappa, F(\varkappa, y)) - d(\varkappa, \varkappa_{n+1}) + d(y, F(y, \varkappa)) - d(y, y_{n+1})\right]\right)$$
$$\leq \varphi\left(\frac{1}{2}\left[d(F(\varkappa_n, y_n), F(\varkappa, y)) + d(F(y_n, \varkappa_n), F(y, \varkappa))\right]\right)$$
$$\leq \psi\left(\frac{d(\varkappa_n, \varkappa) + d(y_n, y)}{2}\right).$$

Now, on taking $n \to \infty$ in the last inequality and using the lower semi-continuity of φ, we obtain

$$\varphi\left(\frac{1}{2}\left[d(\varkappa, F(\varkappa, y)) + d(y, F(y, \varkappa))\right]\right)$$
$$\leq \limsup_{n\to\infty} \varphi\left(\frac{1}{2}\left[d(\varkappa, F(\varkappa, y)) - d(\varkappa, \varkappa_{n+1}) + d(y, F(y, \varkappa)) - d(y, y_{n+1})\right]\right)$$
$$\leq \limsup_{n\to\infty} \psi\left(\frac{d(\varkappa_n, \varkappa) + d(y_n, y)}{2}\right) = 0.$$

Therefore, we get $\varkappa = F(\varkappa, y)$ and $y = F(y, \varkappa)$. Hence, F has a coupled fixed point in X.

Remark 3.3.1. (i) Substituting $\varphi(\varkappa) - \psi(\varkappa)$ for $\psi(\varkappa)$ in Theorem 3.3.1, we can obtain Theorem 3.1.2 (Berinde [150]).

(ii) Considering $\varphi(\varkappa) = \frac{\varkappa}{2}$ and $\psi(\varkappa) = \frac{k\varkappa}{2}$, where $0 \leq k < 1$, in Theorem 3.3.1, we can obtain an analogue of Theorem 3.1.1 (Berinde [149]).

The following example furnishes that the contractive condition (3.3.1) of Theorem 3.3.1 weakens conditions (2.1.14) of Theorem 2.1.14 and (2.1.18) of Theorem 2.1.19, which implies that Theorem 3.3.1 is more general than Theorem 2.1.14 (Bhaskar and Lakshmikantham [55]) and Theorem 2.1.19 (Luong and Thuan [67]), respectively.

Example 3.3.1. Let $X = \mathbb{R}$, then, $(X, \preccurlyeq, d)$ is a POCMS, with partial ordering $\preccurlyeq$ being the usual ordering $\leq$ of real numbers and $d: X \times X \to \mathbb{R}^+$ defined by $d(\varkappa, y) = |\varkappa - y|$ for $\varkappa, y \in X$. Let $F: X \times X \to X$ be defined by $F(\varkappa, y) = \frac{\varkappa - 4y}{8}$ for $\varkappa, y \in X$. Then, F is continuous, has MMP and satisfies the condition (3.3.1) but does not satisfy any of the conditions (2.1.14) and (2.1.18), so that Theorems 2.1.14 and 2.1.19 do not hold here.

Let there exists some $k \in [0, 1)$ such that (2.1.14) holds, so that for $\varkappa \geq u$ and $y \leq v$, we shall have

$$d(F(\varkappa, y), F(u, v)) \leq \frac{k}{2}\,[d(\varkappa, u) + d(y, v)],$$

that is,
$$\left|\frac{\varkappa-4y}{8} - \frac{u-4v}{8}\right| \leq \frac{k}{2}\,[|\varkappa - u| + |y - v|],$$

from which, for $\varkappa = u$, we can obtain $|y - v| \leq k\,|y - v|$, $y \leq v$, which for $y < v$ implies that $1 \leq k$, a contradiction, since $k \in [0, 1)$. Therefore, F does not satisfy (2.1.14). Now, as in Example 3.2.2, it is easy to obtain that the condition (2.1.18) is also not satisfied.

Next, we shall prove that (3.3.1) holds. For, $\varkappa \geq u$ and $y \leq v$, we have

$$\left|\frac{\varkappa-4y}{8} - \frac{u-4v}{8}\right| \leq \frac{1}{8}|\varkappa - u| + \frac{1}{2}|y - v| \text{ and } \left|\frac{y-4\varkappa}{8} - \frac{v-4u}{8}\right| \leq \frac{1}{8}|y - v| + \frac{1}{2}|\varkappa - u|.$$

Adding the last two inequalities, we can exactly obtain (3.3.1) with $\varphi(t) = \frac{1}{2}t$, $\psi(t) = \frac{5}{16}t$. Further, $\varkappa_0\ (= -1)$, $y_0\ (= 1) \in X$ such that the property (P1) holds. Applying Theorem 3.3.1, we can obtain that F has a coupled fixed point $(0, 0)$ in X. But Theorems 2.1.14 and 2.1.19 cannot be applied to F in this example.

Uniqueness Of Coupled Fixed Point

We now prove the uniqueness of the coupled fixed point obtained under the hypotheses of Theorem 3.3.1, by assuming the following additional hypothesis:

Assumption 3.3.1 ([41, 55]). "For every $(\varkappa, y)$, $(\varkappa^*, y^*)$ in $\mathfrak{X} \times \mathfrak{X}$, there exists a (u, v) in $\mathfrak{X} \times \mathfrak{X}$ that is comparable to $(\varkappa, y)$ and $(\varkappa^*, y^*)$".

Theorem 3.3.2. In addition to the hypotheses of Theorem 3.3.1, assume that the Assumption 3.3.1 holds. Then, F has a unique coupled fixed point in $\mathfrak{X}$.

Proof. By Theorem 3.3.1, the set of coupled fixed points of F is non-empty. Suppose that $(\varkappa, y)$ and $(\varkappa^*, y^*)$ be the coupled fixed points of F.

We show that $\varkappa = \varkappa^*$ and $y = y^*$.

By Assumption 3.3.1, there exists some $(u, v) \in \mathfrak{X} \times \mathfrak{X}$ which is comparable to $(\varkappa, y)$ and $(\varkappa^*, y^*)$. Let us define the sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = \mathrm{F}(u_n, v_n), \; v_{n+1} = \mathrm{F}(v_n, u_n), \text{ for } n \geq 0.$$

Since (u, v) is comparable to $(\varkappa, y)$, we assume that $(\varkappa, y) \succcurlyeq (u, v) = (u_0, v_0)$. Now, as in the proof of Theorem 3.3.1, inductively, we can obtain that

$$(\varkappa, y) \succcurlyeq (u_n, v_n) \text{ for } n \geq 0, \tag{3.3.13}$$

therefore, by (3.3.1), we obtain

$$\varphi\left(\frac{d(\varkappa,u_{n+1}) + d(y,v_{n+1})}{2}\right) = \varphi\left(\frac{d(\mathrm{F}(\varkappa,y), \mathrm{F}(u_n,v_n)) + d(\mathrm{F}(y,\varkappa), \mathrm{F}(v_n,u_n))}{2}\right)$$
$$\leq \psi\left(\frac{d(\varkappa,u_n) + d(y,v_n)}{2}\right), \tag{3.3.14}$$

that is, $\varphi(\mathrm{d}_{n+1}) \leq \psi(\mathrm{d}_n)$, where $\mathrm{d}_n = \frac{d(\varkappa,u_n) + d(y,v_n)}{2}$. Now, as in the proof of Theorem 3.3.1, we can obtain that $\{\mathrm{d}_n\}$ converges to some $\mathrm{d} \geq 0$. If $\mathrm{d} > 0$, then we have $\varphi(\mathrm{d}) \leq \limsup_{n\to\infty} \varphi(\mathrm{d}_{n+1}) \leq \limsup_{n\to\infty} \psi(\mathrm{d}_n) < \varphi(\mathrm{d})$, a contradiction. Therefore $\mathrm{d} = 0$, so that $\lim_{n\to\infty} \frac{d(\varkappa,u_n) + d(y,v_n)}{2} = 0$ and hence, we get $\lim_{n\to\infty} d(\varkappa, u_n) = \lim_{n\to\infty} d(y, v_n) = 0$. Similarly, we can obtain that $\lim_{n\to\infty} d(\varkappa^*, u_n) = \lim_{n\to\infty} d(y^*, v_n) = 0$. Now, by uniqueness of limit, we have $\varkappa = \varkappa^*$ and $y = y^*$.

Theorem 3.3.3. In addition to the hypotheses of Theorem 3.3.1 assume that $\varkappa_0, y_0 \in \mathfrak{X}$ are comparable. Then, F has a unique fixed point in $\mathfrak{X}$.

Proof. By Theorem 3.3.1, W.L.O.G., suppose that $\varkappa_0 \preccurlyeq \mathrm{F}(\varkappa_0, y_0)$ and $y_0 \succcurlyeq \mathrm{F}(y_0, \varkappa_0)$. Since $\varkappa_0$ and y_0 are comparable, we have either $\varkappa_0 \preccurlyeq y_0$ or $\varkappa_0 \succcurlyeq y_0$. We consider the second case. Since F has MMP, we get $\varkappa_1 = \mathrm{F}(\varkappa_0, y_0) \succcurlyeq \mathrm{F}(y_0, \varkappa_0) = y_1$. Now, we can obtain inductively that $\varkappa_n \succcurlyeq y_n$ for $n \geq 0$. Also, we have $\varkappa = \lim_{n\to\infty} F(\varkappa_n, y_n)$ and $y = \lim_{n\to\infty} F(y_n, \varkappa_n)$, then, by the continuity of the metric d, we can obtain

$$d(\varkappa, y) = d\left(\lim_{n\to\infty} F(\varkappa_n, y_n), \lim_{n\to\infty} F(y_n, \varkappa_n)\right) = \lim_{n\to\infty} d\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)$$
$$= \lim_{n\to\infty} d(\varkappa_{n+1}, y_{n+1}).$$

Since $\varkappa_n \succcurlyeq y_n$ for $n \geq 0$, by (3.3.1), we have

$$\varphi\left(d\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\right) \leq \psi\big(d(\varkappa_n, y_n)\big), \text{ for } n \geq 0.$$

Now, letting $n \to \infty$ in the last inequality, we obtain

$$\varphi\big(d(\varkappa, y)\big) \leq \limsup_{n\to\infty} \varphi\left(d\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\right) \leq \limsup_{n\to\infty} \psi\big(d(\varkappa_n, y_n)\big).$$

If $\varkappa \neq y$, then using (ψ_i), we obtain that $\varphi\big(d(\varkappa, y)\big) < \varphi\big(d(\varkappa, y)\big)$, a contradiction. Therefore $\varkappa = y$, so that we have $\varkappa = F(\varkappa, \varkappa)$. In a similar way, uniqueness of $\varkappa$ can be achieved.

3.4 COUPLED FIXED POINTS FOR SYMMETRIC (ϕ, ψ) – WEAKLY CONTRACTIVE CONDITION IN PARTIAL METRIC SPACES

In this section, we introduce the notion of symmetric (ϕ, ψ) – weakly contractive condition in POPMS and utilize it to extend the result of Berinde [150] (that is, Theorem 3.1.2) to the partial metric spaces.

We first define the following notion and then, give our result:

Definition 3.4.1. Let (X, $\preccurlyeq$, $\mathfrak{p}$) be a POPMS. Then, the mapping $F: X \times X \to X$ is said to satisfy **symmetric (ϕ, ψ) – weakly contractive condition**, if there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that for all $\varkappa, y, u, v \in X$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$ (or $\varkappa \preccurlyeq u$ and $y \succcurlyeq v$), we have

$$\phi\left(\frac{\mathfrak{p}\big(F(\varkappa,y), F(u,v)\big) + \mathfrak{p}\big(F(y,\varkappa), F(v,u)\big)}{2}\right) \leq \phi\left(\frac{\mathfrak{p}(\varkappa,u)+\mathfrak{p}(y,v)}{2}\right) - \psi\left(\frac{\mathfrak{p}(\varkappa,u)+\mathfrak{p}(y,v)}{2}\right). \qquad (3.4.1)$$

Theorem 3.4.1. Let (X, $\preccurlyeq$, $\mathfrak{p}$) be a POCPMS and $F: X \times X \to X$ be a mapping with MMP on X and there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that F satisfies symmetric (ϕ, ψ) – weakly contractive condition.

Suppose either

(a) F is continuous, or (b) X assumes Assumption 2.1.7.

If X has the property (P3), then F has a coupled fixed point in X.

Proof. Since $\mathfrak{X}$ has property (P3), W.L.O.G., let there exist $\varkappa_0$, $y_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq \mathrm{F}(\varkappa_0, y_0)$ and $y_0 \succcurlyeq \mathrm{F}(y_0, \varkappa_0)$.Then, as in the proof of Theorem 3.3.1, we can easily construct sequences $\{\varkappa_n\}$ and $\{y_n\}$ in $\mathfrak{X}$ such that

$$\varkappa_{n+1} = \mathrm{F}(\varkappa_n, y_n), \quad y_{n+1} = \mathrm{F}(y_n, \varkappa_n)$$

and $\varkappa_n \preccurlyeq \varkappa_{n+1}$, $y_n \succcurlyeq y_{n+1}$, for all $n \geq 0$ holds.

Also, suppose either $\varkappa_{n+1} = \mathrm{F}(\varkappa_n, y_n) \neq \varkappa_n$ or $y_{n+1} = \mathrm{F}(y_n, \varkappa_n) \neq y_n$, otherwise, F has a coupled fixed point and the result holds trivially.

Since $\varkappa_n \preccurlyeq \varkappa_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for $n \geq 0$, on applying the inequality (3.4.1), we have

$$\phi\left(\frac{\mathrm{þ}(\varkappa_{n+1},\varkappa_{n+2}) + \mathrm{þ}(y_{n+1},y_{n+2})}{2}\right) = \phi\left(\frac{\mathrm{þ}\big(\mathrm{F}(\varkappa_n,y_n), \mathrm{F}(\varkappa_{n+1},y_{n+1})\big) + \mathrm{þ}\big(\mathrm{F}(y_n,\varkappa_n), \mathrm{F}(y_{n+1},\varkappa_{n+1})\big)}{2}\right)$$

$$\leq \phi\left(\frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2}\right) - \psi\left(\frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2}\right) \tag{3.4.2}$$

$$\leq \phi\left(\frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2}\right),$$

which implies, on using the condition (i_φ) that

$$\frac{\mathrm{þ}(\varkappa_{n+1},\varkappa_{n+2}) + \mathrm{þ}(y_{n+1},y_{n+2})}{2} \leq \frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2},$$

so that, $\{p_n\}$ is a non-increasing sequence, where $p_n = \frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2} \geq 0$. Thus, there exists some $p \geq 0$ such that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2} = p. \tag{3.4.3}$$

We claim that $p = 0$. On the contrary, assume that $p > 0$. Now, taking $n \to \infty$ in (3.4.2), we get

$$\phi(p) = \lim_{n\to\infty} \phi(p_{n+1}) \leq \lim_{n\to\infty} \phi(p_n) - \lim_{n\to\infty} \psi(p_n) = \phi(p) - \lim_{p_n \to p^+} \psi(p_n) < \phi(p),$$

a contradiction. Therefore, $p = 0$ and hence, we get

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{\mathrm{þ}(\varkappa_n,\varkappa_{n+1}) + \mathrm{þ}(y_n,y_{n+1})}{2} = 0. \tag{3.4.4}$$

Next, we claim that $\{\varkappa_n\}$ and $\{y_n\}$ are Cauchy sequences in $(\mathfrak{X}, \mathrm{þ})$. For, we first show that

$$\lim_{n,m\to\infty} \frac{\mathrm{þ}(\varkappa_n,\varkappa_m) + \mathrm{þ}(y_n,y_m)}{2} = 0. \tag{3.4.5}$$

Let us assume the contrary. So, there exists some $\varepsilon > 0$, for which we can find the sub-sequences $\{\varkappa_{m(j)}\}$, $\{\varkappa_{n(j)}\}$ of $\{\varkappa_n\}$ and $\{y_{m(j)}\}$, $\{y_{n(j)}\}$ of $\{y_n\}$ with $n(j)$ being the smallest index for which

$$n(j) > m(j) > j, \qquad \frac{\mathrm{þ}(\varkappa_{m(j)},\varkappa_{n(j)}) + \mathrm{þ}(y_{m(j)},y_{n(j)})}{2} \geq \varepsilon. \tag{3.4.6}$$

This means

$$\frac{þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(y_{m(j)},y_{n(j)-1})}{2}<\varepsilon. \quad (3.4.7)$$

By (3.4.7), we have

$$\frac{þ(\varkappa_{m(j)},\varkappa_{n(j)})+þ(y_{m(j)},y_{n(j)})}{2}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(þ(\varkappa_{m(j)},\varkappa_{m(j)+1})+þ(\varkappa_{m(j)+1},\varkappa_{n(j)})-þ(\varkappa_{m(j)+1},\varkappa_{m(j)+1})\right)\\ +\left(þ(y_{m(j)},y_{m(j)+1})+þ(y_{m(j)+1},y_{n(j)})-þ(y_{m(j)+1},y_{m(j)+1})\right)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(þ(\varkappa_{m(j)},\varkappa_{m(j)+1})+þ(\varkappa_{m(j)+1},\varkappa_{n(j)})\right)\\ +\left(þ(y_{m(j)},y_{m(j)+1})+þ(y_{m(j)+1},y_{n(j)})\right)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(þ(\varkappa_{m(j)},\varkappa_{m(j)+1})+þ(\varkappa_{m(j)+1},\varkappa_{m(j)})+þ(\varkappa_{m(j)},\varkappa_{n(j)})-þ(\varkappa_{m(j)},\varkappa_{m(j)})\right)\\ +\left(þ(y_{m(j)},y_{m(j)+1})+þ(y_{m(j)+1},y_{m(j)})+þ(y_{m(j)},y_{n(j)})-þ(y_{m(j)},y_{m(j)})\right)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(2þ(\varkappa_{m(j)+1},\varkappa_{m(j)})+þ(\varkappa_{m(j)},\varkappa_{n(j)})\right)\\ +\left(2þ(y_{m(j)+1},y_{m(j)})+þ(y_{m(j)},y_{n(j)})\right)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(\begin{array}{c}2þ(\varkappa_{m(j)+1},\varkappa_{m(j)})+þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(\varkappa_{n(j)-1},\varkappa_{n(j)})\\ -þ(\varkappa_{n(j)-1},\varkappa_{n(j)-1})\end{array}\right)\\ +\left(\begin{array}{c}2þ(y_{m(j)+1},y_{m(j)})+þ(y_{m(j)},y_{n(j)-1})+þ(y_{n(j)-1},y_{n(j)})\\ -þ(y_{n(j)-1},y_{n(j)-1})\end{array}\right)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\left(2þ(\varkappa_{m(j)+1},\varkappa_{m(j)})+þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(\varkappa_{n(j)-1},\varkappa_{n(j)})\right)\\ +\left(2þ(y_{m(j)+1},y_{m(j)})+þ(y_{m(j)},y_{n(j)-1})+þ(y_{n(j)-1},y_{n(j)})\right)\end{array}\right\}$$

$$<2\left\{\frac{þ(\varkappa_{m(j)+1},\varkappa_{m(j)})+þ(y_{m(j)+1},y_{m(j)})}{2}\right\}+\varepsilon+\frac{þ(\varkappa_{n(j)-1},\varkappa_{n(j)})+þ(y_{n(j)-1},y_{n(j)})}{2}. \quad (3.4.8)$$

Letting $j \to \infty$ in (3.4.8) and then using (3.4.4) and (3.4.6), we get

$$\lim_{j\to\infty}\frac{þ(\varkappa_{m(j)},\varkappa_{n(j)})+þ(y_{m(j)},y_{n(j)})}{2}=\varepsilon. \quad (3.4.9)$$

Also,

$$þ(\varkappa_{m(j)},\varkappa_{n(j)})\leq þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(\varkappa_{n(j)-1},\varkappa_{n(j)}),$$

$$þ(y_{m(j)},y_{n(j)})\leq þ(y_{m(j)},y_{n(j)-1})+þ(y_{n(j)-1},y_{n(j)}).$$

Then, we get

$$þ(\varkappa_{m(j)},\varkappa_{n(j)})+þ(y_{m(j)},y_{n(j)})\leq\{þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(y_{m(j)},y_{n(j)-1})\}$$
$$+\{þ(\varkappa_{n(j)-1},\varkappa_{n(j)})+þ(y_{n(j)-1},y_{n(j)})\}. \quad (3.4.10)$$

Similarly, we have

$$þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(y_{m(j)},y_{n(j)-1})\leq\{þ(\varkappa_{m(j)},\varkappa_{n(j)})+þ(y_{m(j)},y_{n(j)})\}$$

$$+\{þ(\varkappa_{n(j)},\varkappa_{n(j)-1})+þ(y_{n(j)},y_{n(j)-1})\}. \quad (3.4.11)$$

Taking j → ∞ in (3.4.10) and (3.4.11) and using (3.4.4), (3.4.9), we get

$$\lim_{j\to\infty}\frac{þ(\varkappa_{m(j)},\varkappa_{n(j)-1})+þ(y_{m(j)},y_{n(j)-1})}{2}=\varepsilon. \quad (3.4.12)$$

Now, since $\varkappa_{m(j)} \preccurlyeq \varkappa_{n(j)-1}$ and $y_{m(j)} \succcurlyeq y_{n(j)-1}$, using (3.4.1), we have

$$\phi\left(\frac{þ(\varkappa_{n(j)},\varkappa_{m(j)+1})+þ(y_{n(j)},y_{m(j)+1})}{2}\right)$$

$$=\phi\left(\frac{þ\big(F(\varkappa_{n(j)-1},y_{n(j)-1}),F(\varkappa_{m(j)},y_{m(j)})\big)+þ\big(F(y_{n(j)-1},\varkappa_{n(j)-1}),F(y_{m(j)},\varkappa_{m(j)})\big)}{2}\right)$$

$$\le\phi\left(\frac{þ(\varkappa_{n(j)-1},\varkappa_{m(j)})+þ(y_{n(j)-1},y_{m(j)})}{2}\right)-\psi\left(\frac{þ(\varkappa_{n(j)-1},\varkappa_{m(j)})+þ(y_{n(j)-1},y_{m(j)})}{2}\right).$$

Taking j → ∞ in the above inequality, then using (3.4.12) and the properties of ϕ and ψ, we obtain

$$\phi(\varepsilon)\le\phi(\varepsilon)-\lim_{j\to\infty}\psi\left(\frac{þ(\varkappa_{n(j)-1},\varkappa_{m(j)})+þ(y_{n(j)-1},y_{m(j)})}{2}\right)<\phi(\varepsilon),$$

a contradiction. Hence, (3.4.5) holds and we have

$$\lim_{n,m\to\infty}þ(\varkappa_n,\varkappa_m)=0 \quad \text{and} \quad \lim_{n,m\to\infty}þ(y_n,y_m)=0. \quad (3.4.13)$$

Now, by (2.2.1), we get

$$þ^s(\varkappa_n,\varkappa_m)\le 2\,þ(\varkappa_n,\varkappa_m) \quad \text{and} \quad þ^s(y_n,y_m)\le 2\,þ(y_n,y_m). \quad (3.4.14)$$

On taking n, m → ∞ in (3.4.14) and using (3.4.13), we obtain that

$$\lim_{n,m\to\infty}þ^s(\varkappa_n,\varkappa_m)=0 \quad \text{and} \quad \lim_{n,m\to\infty}þ^s(y_n,y_m)=0. \quad (3.4.15)$$

Therefore, $\{\varkappa_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (Ҳ, $þ^s$). Also, since the space (Ҳ, þ) is complete, the space (Ҳ, $þ^s$) is also complete. Therefore, there exist some $\varkappa$, y ∈ Ҳ such that

$$\lim_{n\to\infty}þ^s(\varkappa_n,\varkappa)=0 \quad \text{and} \quad \lim_{n\to\infty}þ^s(y_n,y)=0. \quad (3.4.16)$$

Again using (2.2.1), we have $þ^s(\varkappa_n,\varkappa)=2\,þ(\varkappa_n,\varkappa)-þ(\varkappa_n,\varkappa_n)-þ(\varkappa,\varkappa)$.

On taking n → ∞ in the above equation and using (3.4.16) and (3.4.13), we obtain

$$\lim_{n\to\infty}þ(\varkappa_n,\varkappa)=\frac{1}{2}\,þ(\varkappa,\varkappa). \quad (3.4.17)$$

Also, we have $þ(\varkappa,\varkappa)\le þ(\varkappa,\varkappa_n)$ for all n ∈ ℕ. Then, on taking n → ∞, we have

$$þ(\varkappa,\varkappa)\le\lim_{n\to\infty}þ(\varkappa,\varkappa_n). \quad (3.4.18)$$

Now, using (3.4.17) and (3.4.18), we can obtain $\lim_{n\to\infty}þ(\varkappa,\varkappa_n)=þ(\varkappa,\varkappa)=0$.

Similarly, we can obtain that $\lim_{n\to\infty}þ(y_n,y)=þ(y,y)=0$.

Therefore, we get

$$\lim_{n\to\infty} þ(\varkappa_n, \varkappa) = þ(\varkappa, \varkappa) = 0 \quad \text{and} \quad \lim_{n\to\infty} þ(y_n, y) = þ(y, y) = 0. \qquad (3.4.19)$$

Also, by þ2, we obtain $0 \le þ(\varkappa_n, \varkappa_n) \le þ(\varkappa_n, \varkappa)$ and $0 \le þ(y_n, y_n) \le þ(y_n, y)$ for all $n \in \mathbb{N}$. On taking $n \to \infty$ and using (3.4.19), we obtain

$$\begin{aligned} \lim_{n\to\infty} þ(\varkappa_n, \varkappa) &= \lim_{n\to\infty} þ(\varkappa_n, \varkappa_n) = þ(\varkappa, \varkappa) = 0, \\ \lim_{n\to\infty} þ(y_n, y) &= \lim_{n\to\infty} þ(y_n, y_n) = þ(y, y) = 0. \end{aligned} \qquad (3.4.20)$$

We now show that $\varkappa = ₣(\varkappa, y)$ and $y = ₣(y, \varkappa)$.

Let us assume that assumption (a) holds.

We consider the following steps:

Step 1. We show that $þ\big(₣(\varkappa, y), ₣(\varkappa, y)\big) = 0$ and $þ\big(₣(y, \varkappa), ₣(y, \varkappa)\big) = 0$.

Now, since $\varkappa \preccurlyeq \varkappa$ and $y \preccurlyeq y$, using (3.4.1), we get

$$\phi\left(\frac{þ\big(₣(\varkappa,y), ₣(\varkappa,y)\big) + þ\big(₣(y,\varkappa), ₣(y,\varkappa)\big)}{2}\right) \le \phi\left(\frac{þ(\varkappa,\varkappa) + þ(y,y)}{2}\right) - \psi\left(\frac{þ(\varkappa,\varkappa)+þ(y,y)}{2}\right)$$

$$= \phi(0) - \psi(0) = -\psi(0) \le 0,$$

which implies that $\frac{þ\big(₣(\varkappa,y), ₣(\varkappa,y)\big) + þ\big(₣(y,\varkappa), ₣(y,\varkappa)\big)}{2} = 0$, so that $þ\big(₣(\varkappa, y), ₣(\varkappa, y)\big) = 0$ and $þ\big(₣(y, \varkappa), ₣(y, \varkappa)\big) = 0$.

Step 2. We now show the following:

$$\lim_{n\to\infty} þ\big(\varkappa_{n+1}, ₣(\varkappa, y)\big) = þ(₣(\varkappa, y), ₣(\varkappa, y))$$

and $\lim_{n\to\infty} þ\big(y_{n+1}, ₣(y, \varkappa)\big) = þ(₣(y, \varkappa), ₣(y, \varkappa))$.

For, since $\varkappa_{n+1} = ₣(\varkappa_n, y_n)$, we obtain $þ\big(\varkappa_{n+1}, ₣(\varkappa, y)\big) = þ\big(₣(\varkappa_n, y_n), ₣(\varkappa, y)\big)$. Further $\varkappa_n \to \varkappa$ and $y_n \to y$ as $n \to \infty$ in (Ӿ, þ) and ₣ is continuous, then by Lemma 2.2.2, we obtain that $₣(\varkappa_n, y_n) \to ₣(\varkappa, y)$ as $n \to \infty$ in (Ӿ, þ), so that $\lim_{n\to\infty} þ\big(₣(\varkappa_n, y_n), ₣(\varkappa, y)\big) = þ(₣(\varkappa, y), ₣(\varkappa, y)) = 0$. Similarly, we can obtain $\lim_{n\to\infty} þ\big(₣(y_n, \varkappa_n), ₣(y, \varkappa)\big) = þ(₣(y, \varkappa), ₣(y, \varkappa)) = 0$.

Step 3. Finally, we shall show $\varkappa = ₣(\varkappa, y)$ and $y = ₣(y, \varkappa)$.

For, we have

$$\begin{aligned} þ(\varkappa, ₣(\varkappa, y)) &\le þ(\varkappa, \varkappa_{n+1}) + þ(\varkappa_{n+1}, ₣(\varkappa, y)) - þ(\varkappa_{n+1}, \varkappa_{n+1}) \\ &\le þ(\varkappa, \varkappa_{n+1}) + þ(\varkappa_{n+1}, ₣(\varkappa, y)). \end{aligned}$$

Taking $n \to \infty$ in the last inequality, using (3.4.20) and Step 2, we get $þ(\varkappa, ₣(\varkappa, y)) = 0$. Therefore, we have $\varkappa = ₣(\varkappa, y)$. Similarly, we can obtain $y = ₣(y, \varkappa)$.

Next, assume that assumption (b) holds.

Since $\varkappa_n \preccurlyeq \varkappa_{n+1}$, $y_n \succcurlyeq y_{n+1}$ and using (3.4.20), we obtain that $\{\varkappa_n\}$ is a non-decreasing sequence converging to $\varkappa$ in (Ӿ, þ) and $\{y_n\}$ is a non-increasing sequence converging to y in (Ӿ, þ). Therefore, using the assumption (b), for all n ≥ 0, we obtain that

$$\varkappa_n \preccurlyeq \varkappa \text{ and } y \preccurlyeq y_n. \tag{3.4.21}$$

Then by (3.4.1), we have

$$\phi\left(\frac{þ(\varkappa_{n+1}, Ꞙ(\varkappa,y)) + þ(y_{n+1}, Ꞙ(y,\varkappa))}{2}\right) = \phi\left(\frac{þ(Ꞙ(\varkappa_n,y_n), Ꞙ(\varkappa,y)) + þ(Ꞙ(y_n,\varkappa_n), Ꞙ(y,\varkappa))}{2}\right)$$
$$\leq \phi\left(\frac{þ(\varkappa_n,\varkappa) + þ(y_n,y)}{2}\right) - \psi\left(\frac{þ(\varkappa_n,\varkappa) + þ(y_n,y)}{2}\right).$$

Taking n → ∞ in the last inequality, using (3.4.20) and the properties of ϕ and ψ, we obtain that

$$\lim_{n\to\infty} þ(\varkappa_{n+1}, Ꞙ(\varkappa,y)) = 0 \quad \text{and} \quad \lim_{n\to\infty} þ(y_{n+1}, Ꞙ(y,\varkappa)) = 0. \tag{3.4.22}$$

Also, we have

$$þ(\varkappa, Ꞙ(\varkappa, y)) \leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1}, Ꞙ(\varkappa,y)) - þ(\varkappa_{n+1},\varkappa_{n+1})$$
$$\leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1}, Ꞙ(\varkappa,y)).$$

Taking n → ∞ in the last inequality, using (3.4.20) and (3.4.22), we get $þ(\varkappa, Ꞙ(\varkappa, y)) = 0$, so that $\varkappa = Ꞙ(\varkappa, y)$. Similarly, we can get $y = Ꞙ(y, \varkappa)$. Therefore, $(\varkappa, y)$ is a coupled fixed point of Ꞙ.

Example 3.4.1. Let Ӿ = ℝ, equipped with the partial metric þ given by $þ(\varkappa, y) = \max\{\varkappa, y\}$ and the natural ordering ≤ of real numbers. Let Ꞙ: Ӿ × Ӿ → Ӿ be defined as $Ꞙ(\varkappa, y) = \frac{\varkappa - y}{8}$ for $\varkappa$, y ∈ Ӿ. Then, Ꞙ has the MMP on Ӿ. We next show that Ꞙ satisfies the condition (3.4.1). For,

$$þ(Ꞙ(\varkappa, y), Ꞙ(u, v)) = \max\left\{\frac{|\varkappa - y|}{8}, \frac{|u-v|}{8}\right\} = \frac{1}{8}\max\{\varkappa - y, y - \varkappa, u - v, v - u\}$$
$$= \frac{1}{8}\max\{\varkappa, y, u, v\} \leq \frac{1}{8}\max\{\varkappa, u\} + \frac{1}{8}\max\{y, v\}.$$

Similarly, we can get $þ(Ꞙ(y, \varkappa), Ꞙ(v, u)) \leq \frac{1}{8}\max\{\varkappa, u\} + \frac{1}{8}\max\{y, v\}$.

Adding the last two inequalities, we can obtain

$$þ\big(Ꞙ(\varkappa,y), Ꞙ(u,v)\big) + þ\big(Ꞙ(y,\varkappa), Ꞙ(v,u)\big) \leq \frac{þ(\varkappa,u) + þ(y,v)}{8} + \frac{þ(\varkappa,u) + þ(y,v)}{8},$$

or

$$þ\big(Ꞙ(\varkappa,y), Ꞙ(u,v)\big) + þ\big(Ꞙ(y,\varkappa), Ꞙ(v,u)\big) \leq \frac{þ(\varkappa,u) + þ(y,v)}{2} - \frac{1}{2}\frac{þ(\varkappa,u) + þ(y,v)}{2},$$

therefore, the condition (3.4.1) holds for $\phi(\mathfrak{t}) = \mathfrak{t}/2$ and $\psi(\mathfrak{t}) = 3\mathfrak{t}/8$. Further, the other conditions of Theorem 3.4.1 are also satisfied so that (0, 0) is a coupled fixed point of Ꞙ.

Remark 3.4.1. Theorem 3.4.1 extends Theorem 3.1.2 (Berinde [150]) to the partial metric spaces.

Corollary 3.4.1. Let (Ꭓ, ≼, ϸ) be a POCPMS, Ꞙ: Ꭓ × Ꭓ → Ꭓ be a mapping with MMP on Ꭓ and there exists some ķ ∈ [0, 1) such that ϰ, y, u, v in Ꭓ with ϰ ≽ u and y ≼ v (or ϰ ≼ u and y ≽ v), we have

$$ϸ(Ꞙ(ϰ, y), Ꞙ(u, v)) + ϸ(Ꞙ(y, ϰ), Ꞙ(v, u)) \leq ķ\, [ϸ(ϰ, u) + ϸ(y,v)]. \qquad (3.4.23)$$

Suppose either

(a) Ꞙ is continuous, or (b) Ꭓ assumes Assumption 2.1.7.

If Ꭓ has the property (P3), then Ꞙ has a coupled fixed point in Ꭓ.

Proof. Considering $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$ and $\psi(\mathfrak{t}) = (1 - ķ)\frac{\mathfrak{t}}{2}$, $0 \leq ķ < 1$ in Theorem 3.4.1, we can obtain the required result.

Uniqueness Of Coupled Fixed Point

Now, we establish the uniqueness of the coupled fixed point obtained under the hypotheses of Theorem 3.4.1.

Theorem 3.4.2. In addition to the hypotheses of Theorem 3.4.1, assume that Assumption 3.3.1 also holds. Then, Ꞙ has a unique coupled fixed point in Ꭓ.

Proof. By Theorem 3.4.1, the set of coupled fixed points of Ꞙ is non-empty. To prove the result, we shall show that if (ϰ, y) and (ϰ*, y*) be the two coupled fixed points of Ꞙ, then

$$ϸ(ϰ, ϰ^*) = 0 \quad \text{and} \quad ϸ(y, y^*) = 0.$$

By Assumption 3.3.1, there exists some (u, v) ∈ Ꭓ × Ꭓ which is comparable to (ϰ, y) and (ϰ*, y*). Let us define two sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$u_0 = u,\ \ v_0 = v,\ \ u_{n+1} = Ꞙ(u_n, v_n),\ \ v_n = Ꞙ(v_n, u_n), \text{ for } n \geq 0.$$

Since (u, v) is comparable to (ϰ, y), we suppose that (ϰ, y) ≽ (u, v) = (u_0, v_0).

Now, as in the proof of Theorem 3.4.1, inductively, we can obtain that

$$(ϰ, y) \succcurlyeq (u_n, v_n) \text{ for } n \geq 0. \qquad (3.4.24)$$

Then, by (3.4.1), we get

$$\phi\left(\frac{ϸ(ϰ,u_{n+1}) + ϸ(y,v_{n+1})}{2}\right) = \phi\left(\frac{ϸ(Ꞙ(ϰ,y), Ꞙ(u_n,v_n)) + ϸ(Ꞙ(y, ϰ), Ꞙ(v_n,u_n))}{2}\right)$$

$$\leq \phi\left(\frac{ϸ(ϰ,u_n) + ϸ(y,v_n)}{2}\right) - \psi\left(\frac{ϸ(ϰ,u_n) + ϸ(y,v_n)}{2}\right). \qquad (3.4.25)$$

Now, since ψ is a non-negative function, by (3.4.25), we get

$$\phi\left(\frac{þ(\varkappa,u_{n+1}) + þ(y,v_{n+1})}{2}\right) \leq \phi\left(\frac{þ(\varkappa,u_n) + þ(y,v_n)}{2}\right),$$

then, by monotone property of ϕ, it follows that $\{\beta_n\}$ with $\beta_n = \frac{þ(\varkappa,u_n) + þ(y,v_n)}{2}$, $n \geq 0$, is a non-increasing sequence. So, there exists some $\beta \geq 0$ such that $\lim_{n\to\infty}\beta_n = \beta$. We claim that $\beta = 0$. On the contrary, assume that $\beta > 0$. Taking $n \to \infty$ in (3.4.25), we obtain

$$\phi(\beta) \leq \phi(\beta) - \lim_{n\to\infty}\psi(\beta_n) = \phi(\beta) - \lim_{\alpha_n\to\alpha}\psi(\beta_n) < \phi(\beta),$$

a contradiction. Therefore, $\beta = 0$, so that $\lim_{n\to\infty}\frac{þ(\varkappa,u_n) + þ(y,v_n)}{2} = 0$ and hence, we can obtain that $\lim_{n\to\infty}þ(\varkappa,u_n) = \lim_{n\to\infty}þ(y,v_n) = 0$. Similarly, we have $\lim_{n\to\infty}þ(\varkappa^*,u_n) = \lim_{n\to\infty}þ(y^*,v_n) = 0$.

By þ4, we have $þ(\varkappa, \varkappa^*) \leq þ(\varkappa,u_n) + þ(u_n,\varkappa^*) - þ(u_n,u_n)$

$$\leq þ(\varkappa,u_n) + þ(u_n,\varkappa^*),$$

then, on taking $n \to \infty$, we get $þ(\varkappa, \varkappa^*) = 0$. Similarly, we have $þ(y, y^*) = 0$. Therefore, $\varkappa = \varkappa^*$ and $y = y^*$. Thus, the result is proved.

Theorem 3.4.3. In addition to the hypotheses of Theorem 3.4.1, assume that $\varkappa_0, y_0 \in X$ are comparable. Then, Ꞙ has a unique fixed point in X.

Proof. To prove the result, we show that $\varkappa = y$, if $(\varkappa, y)$ is a coupled fixed point of Ꞙ. On the contrary, suppose $\varkappa \neq y$. By Theorem 3.4.1, W.L.O.G., assume that $\varkappa_0 \preccurlyeq Ꞙ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ꞙ(y_0, \varkappa_0)$. Now, since $\varkappa_0, y_0$ are comparable, we have $\varkappa_0 \preccurlyeq y_0$ or $\varkappa_0 \succcurlyeq y_0$. W.L.O.G., suppose that $\varkappa_0 \succcurlyeq y_0$. Also, since Ꞙ has the MMP, we have $\varkappa_1 = Ꞙ(\varkappa_0, y_0) \succcurlyeq Ꞙ(y_0, \varkappa_0) = y_1$. Now, inductively, we can obtain that $\varkappa_n \succcurlyeq y_n$, for $n \geq 0$. Also, $\lim_{n\to\infty}þ(\varkappa,\varkappa_n) = 0$ and $\lim_{n\to\infty}þ(y,y_n) = 0$.

Now, on repeatedly applying the properties of partial metric, we get

$$\begin{aligned} þ(\varkappa, y) &\leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1},y) - þ(\varkappa_{n+1},\varkappa_{n+1}) \\ &\leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1},y) \\ &\leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1},y_{n+1}) + þ(y_{n+1},y) - þ(y_{n+1},y_{n+1}) \\ &\leq þ(\varkappa,\varkappa_{n+1}) + þ(\varkappa_{n+1},y_{n+1}) + þ(y_{n+1},y) \\ &= þ(\varkappa,\varkappa_{n+1}) + þ\big(Ꞙ(\varkappa_n,y_n), Ꞙ(y_n,\varkappa_n)\big) + þ(y_{n+1},y), \end{aligned}$$

then, using the monotone property of ϕ and the property (iii_φ), we have

$$\phi(þ(\varkappa,y)) \leq \phi\left(þ(\varkappa,\varkappa_{n+1}) + þ\big(Ꞙ(\varkappa_n,y_n), Ꞙ(y_n,\varkappa_n)\big) + þ(y_{n+1},y)\right)$$

$$\leq \phi\big(\text{þ}(\varkappa, \varkappa_{n+1})\big) + \phi\left(\text{þ}\big(\text{Ғ}(\varkappa_n, y_n), \text{Ғ}(y_n, \varkappa_n)\big)\right) + \phi\big(\text{þ}(y_{n+1}, y)\big)$$

$$\leq \phi\big(\text{þ}(\varkappa, \varkappa_{n+1})\big) + \phi\big(\text{þ}(\varkappa_n, y_n)\big) - \psi\big(\text{þ}(\varkappa_n, y_n)\big) + \phi\big(\text{þ}(y_{n+1}, y)\big),$$

then, on taking $n \to \infty$ and using the properties of ϕ and ψ, we get

$$\phi(\text{þ}(\varkappa, y)) \leq \phi(0) + \phi(0) - \lim_{n\to\infty}\psi\big(\text{þ}(\varkappa_n, y_n)\big) + \phi(0)$$

$$= -\lim_{n\to\infty}\psi\big(\text{þ}(\varkappa_n, y_n)\big).$$

We now consider the following cases:

Case 1. If $\lim_{n\to\infty}\text{þ}(\varkappa_n, y_n) > 0$, then $\lim_{n\to\infty}\psi\big(\text{þ}(\varkappa_n, y_n)\big) > 0$, so that we have $\phi(\text{þ}(\varkappa, y)) < 0$, a contradiction.

Case 2. If $\lim_{n\to\infty}\text{þ}(\varkappa_n, y_n) = 0$, then $\lim_{n\to\infty}\psi\big(\text{þ}(\varkappa_n, y_n)\big) = 0$, so we have $\phi(\text{þ}(\varkappa, y)) \leq 0$.

Subcase (i). If $\phi(\text{þ}(\varkappa, y)) < 0$, a contradiction.

Subcase (ii). If $\phi(\text{þ}(\varkappa, y)) = 0$, then we have $\text{þ}(\varkappa, y) = 0$, so that $\varkappa = y$, a contradiction, since we have $\varkappa \neq y$.

Therefore, in each of the above case, we get a contradiction. Hence, the assumption $\varkappa \neq y$ is wrong. Thus, we have $\varkappa = y$.

3.5. APPLICATIONS

This section consists of the applications of the results proved in sections 3.3 and 3.4.

First, as an application of the results proved in section 3.3, we study the existence of the unique solution of the following integral equation:

$$\varkappa(\mathfrak{t}) = \int_c^d \big(\text{K}_1(s, \mathfrak{t}) - \text{K}_2(s, \mathfrak{t})\big)\left(f_1\big(s, \varkappa(s)\big) + f_2\big(s, \varkappa(s)\big)\right) ds + \text{ђ}(\mathfrak{t}), \ \mathfrak{t} \in I \ (= [c, d]). \tag{3.5.1}$$

Denote by Θ, the class of functions θ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following assumptions:

(i) θ is non-decreasing; (3.5.2)

(ii) there exists some $\psi \in \Psi_\varphi$ such that $\theta(\mathfrak{r}) = \psi\left(\frac{\mathfrak{r}}{2}\right)$ for all $\mathfrak{r} \in \mathbb{R}^+$; (3.5.3)

(iii) $\limsup_{n\to\infty} \theta(z_n) < \alpha\mathfrak{r}$ if $\lim_{n\to\infty} z_n = \mathfrak{r} > 0$ for some $\alpha \in (0, 1)$; (3.5.4)

(iv) $\lim_{n\to\infty}\theta(z_n) = 0$ if $\lim_{n\to\infty} z_n = 0$ for $z_n \in \mathbb{R}^+$. (3.5.5)

Suppose that K_1, K_2, f_1, f_2 fulfil the following assumptions:

Assumption 3.5.1. (i) $\text{K}_1(s, \mathfrak{t}), \text{K}_2(s, \mathfrak{t}) \geq 0$ for all $\mathfrak{t}, s \in I$;

(ii) there exist λ > 0, μ > 0 and $\theta \in \Theta$ such that for all ƫ ∈ I and ϰ, y ∈ ℝ with ϰ ≥ y, we have

$$0 \leq f_1(ƫ, \varkappa) - f_1(ƫ, y) \leq \lambda\, \theta(\varkappa - y) \tag{3.5.6}$$

and

$$-\mu\, \theta(\varkappa - y) \leq f_2(ƫ, \varkappa) - f_2(ƫ, y) \leq 0; \tag{3.5.7}$$

(iii) there exists some $\alpha \in (0, 1)$ satisfying (3.5.4) such that

$$\alpha\beta \leq 1, \tag{3.5.8}$$

where,

$$\beta = (\lambda + \mu) \cdot \sup_{ƫ \in I} \int_c^d \big(K_1(s, ƫ) + K_2(s, ƫ)\big)\, ds. \tag{3.5.9}$$

Definition 3.5.1. An element $(\hat{\varkappa}, \hat{y}) \in Ӽ \times Ӽ$, where $Ӽ = C(I, \mathbb{R})$ is called a **coupled lower and upper solution** of the integral equation (3.5.1) if for all ƫ ∈ I,

$$\hat{\varkappa}(ƫ) \leq \hat{y}(ƫ),$$

$$\hat{\varkappa}(ƫ) \leq \int_c^d K_1(s, ƫ)\left(f_1\big(s, \hat{\varkappa}(s)\big) + f_2\big(s, \hat{y}(s)\big)\right) ds - \int_c^d K_2(s, ƫ)\left(f_1\big(s, \hat{y}(s)\big) + f_2\big(s, \hat{\varkappa}(s)\big)\right) ds + ђ(ƫ)$$

and

$$\hat{y}(ƫ) \geq \int_c^d K_1(s, ƫ)\left(f_1\big(s, \hat{y}(s)\big) + f_2\big(s, \hat{\varkappa}(s)\big)\right) ds - \int_c^d K_2(s, ƫ)\left(f_1\big(s, \hat{\varkappa}(s)\big) + f_2\big(s, \hat{y}(s)\big)\right) ds + ђ(ƫ).$$

Theorem 3.5.1. Consider the integral equation (3.5.1) with $K_i \in C(I \times I, \mathbb{R})$, $f_i \in C(I \times \mathbb{R}, \mathbb{R})$ for i = 1, 2 and ђ ∈ Ӽ (= C(I, ℝ)). Let $(\hat{\varkappa}, \hat{y})$ be a coupled lower-upper solution for (3.5.1) and the Assumption 3.5.1 is satisfied. Then, the integral equation (3.5.1) has a unique solution in Ӽ.

Proof. Consider the following ordering on Ӽ:

for ϰ, y ∈ Ӽ, $\varkappa \preccurlyeq y \Leftrightarrow \varkappa(ƫ) \leq y(ƫ)$, for all ƫ ∈ I.

Also Ӽ is a complete metric space w.r.t. the sup metric

$$ḓ(\varkappa, y) = \sup_{ƫ \in I} |\varkappa(ƫ) - y(ƫ)|, \text{ for } \varkappa, y \in Ӽ.$$

Further, the condition (b) in Theorem 3.3.1 (that is, Assumption 2.1.7) also holds in Ӽ. Also, Ӽ × Ӽ is a poset under the order relation given below:

$(\varkappa, y), (u, v) \in Ӽ \times Ӽ$, $(\varkappa, y) \preccurlyeq (u, v) \Leftrightarrow \varkappa(ƫ) \leq u(ƫ)$ and $y(ƫ) \geq v(ƫ)$, for all ƫ ∈ I.

For ϰ, y ∈ Ӽ, max{ϰ(ƫ), y(ƫ)} and min{ϰ(ƫ), y(ƫ)} for each ƫ ∈ I, are in Ӽ and are upper and lower bounds of ϰ, y, respectively. Hence, for every (ϰ, y), (u, v) ∈ Ӽ × Ӽ, there exists (max{ϰ, u}, min{y, v}) ∈ Ӽ × Ӽ which is comparable to (ϰ, y) and (u, v).

Define the mapping Ꞙ: Ӽ × Ӽ → Ӽ by

$$Ꞙ(\varkappa,y)(ƫ) = \int_c^d K_1(s, ƫ)\left(f_1\big(s, \varkappa(s)\big) + f_2\big(s, y(s)\big)\right) ds$$

$$- \int_c^d K_2(s,\mathfrak{t}) \left(f_1\big(s, y(s)\big) + f_2\big(s, \varkappa(s)\big)\right) ds + \mathfrak{h}(\mathfrak{t}), \quad \text{for all } \mathfrak{t} \in I.$$

We claim that $\mathfrak{F}$ has the MMP.

For, let $\varkappa_1, \varkappa_2 \in \mathfrak{X}$ with $\varkappa_1 \preccurlyeq \varkappa_2$ (that is, $\varkappa_1(\mathfrak{t}) \leq \varkappa_2(\mathfrak{t})$ for all $\mathfrak{t} \in I$).

Then, by Assumption 3.5.1, for any $y \in \mathfrak{X}$ and all $\mathfrak{t} \in I$, we have

$$\mathfrak{F}(\varkappa_1, y)(\mathfrak{t}) - \mathfrak{F}(\varkappa_2, y)(\mathfrak{t}) = \int_c^d K_1(s,\mathfrak{t}) \left(f_1\big(s, \varkappa_1(s)\big) - f_1\big(s, \varkappa_2(s)\big)\right) ds$$

$$- \int_c^d K_2(s,\mathfrak{t}) \left(f_2\big(s, \varkappa_1(s)\big) - f_2\big(s, \varkappa_2(s)\big)\right) ds \leq 0,$$

which implies that $\mathfrak{F}(\varkappa_1, y) \preccurlyeq \mathfrak{F}(\varkappa_2, y)$.

Similarly, if $y_1, y_2 \in \mathfrak{X}$ and $y_1 \preccurlyeq y_2$, then we have $\mathfrak{F}(\varkappa, y_1) \succcurlyeq \mathfrak{F}(\varkappa, y_2)$ for $\varkappa \in \mathfrak{X}$. Let $\alpha \in (0, 1)$ be as mentioned in Assumption 3.5.1. Then, for $\varkappa, y, u, v \in \mathfrak{X}$ such that $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we have

$$\mathfrak{F}(\varkappa, y)(\mathfrak{t}) - \mathfrak{F}(u, v)(\mathfrak{t}) = \begin{Bmatrix} \int_c^d K_1(s,\mathfrak{t}) \left(f_1\big(s, \varkappa(s)\big) + f_2\big(s, y(s)\big)\right) ds \\ - \int_c^d K_2(s,\mathfrak{t}) \left(f_1\big(s, y(s)\big) + f_2\big(s, \varkappa(s)\big)\right) ds + \mathfrak{h}(\mathfrak{t}) \end{Bmatrix}$$

$$- \begin{Bmatrix} \int_c^d K_1(s,\mathfrak{t}) \left(f_1\big(s, u(s)\big) + f_2\big(s, v(s)\big)\right) ds \\ - \int_c^d K_2(s,\mathfrak{t}) \left(f_1\big(s, v(s)\big) + f_2\big(s, u(s)\big)\right) ds + \mathfrak{h}(\mathfrak{t}) \end{Bmatrix}$$

$$= \int_c^d K_1(s,\mathfrak{t}) \left(f_1\big(s, \varkappa(s)\big) - f_1\big(s, u(s)\big) + f_2\big(s, y(s)\big) - f_2\big(s, v(s)\big)\right) ds$$

$$+ \int_c^d K_2(s,\mathfrak{t}) \left(f_1\big(s, v(s)\big) - f_1\big(s, y(s)\big) + f_2\big(s, u(s)\big) - f_2\big(s, \varkappa(s)\big)\right) ds$$

$$\leq \int_c^d K_1(s,\mathfrak{t}) \left[\lambda\theta(\varkappa(s) - u(s)) + \mu\theta(v(s) - y(s))\right] ds$$

$$+ \int_c^d K_2(s,\mathfrak{t}) \left[\lambda\theta(v(s) - y(s)) + \mu\theta(\varkappa(s) - u(s))\right] ds. \tag{3.5.10}$$

Since θ is a non-decreasing function and $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we have

$$\theta(\varkappa(s) - u(s)) \leq \theta(\sup_{\mathfrak{t}\in I}|\varkappa(\mathfrak{t}) - u(\mathfrak{t})|) = \theta\big(\mathfrak{d}(\varkappa, u)\big),$$

and $\theta(v(s) - y(s)) \leq \theta(\sup_{\mathfrak{t}\in I}|v(\mathfrak{t}) - y(\mathfrak{t})|) = \theta\big(\mathfrak{d}(v, y)\big)$,

hence, using (3.5.10), we can obtain that

$$|\mathfrak{F}(\varkappa, y)(\mathfrak{t}) - \mathfrak{F}(u,v)(\mathfrak{t})| \leq \int_c^d K_1(s,\mathfrak{t}) \left[\lambda\theta\big(\mathfrak{d}(\varkappa, u)\big) + \mu\theta\big(\mathfrak{d}(v, y)\big)\right] ds$$

$$+ \int_c^d K_2(s,\mathfrak{t}) \left[\lambda\theta\big(\mathfrak{d}(v, y)\big) + \mu\theta\big(\mathfrak{d}(\varkappa, u)\big)\right] ds. \tag{3.5.11}$$

Similarly, we have

$$|\mathfrak{F}(y, \varkappa)(\mathfrak{t}) - \mathfrak{F}(v, u)(\mathfrak{t})| \leq \int_c^d K_1(s,\mathfrak{t}) \left[\lambda\theta\big(\mathfrak{d}(v, y)\big) + \mu\theta\big(\mathfrak{d}(\varkappa, u)\big)\right] ds$$

$$+ \int_c^d K_2(s,\mathfrak{t}) \left[\lambda\theta\big(\mathfrak{d}(\varkappa, u)\big) + \mu\theta\big(\mathfrak{d}(v, y)\big)\right] ds. \tag{3.5.12}$$

Adding (3.5.11) and (3.5.12), multiplying with α and dividing by 2, then taking supremum w.r.t. ƭ and using (3.5.8) and (3.5.9), we have

$$\alpha\,\frac{ḑ\big(Ғ(\varkappa,y),Ғ(u,v)\big)+ḑ(Ғ(y,\varkappa),Ғ(v,u))}{2}$$

$$\le \alpha(\lambda+\mu)\sup_{ƭ\in I}\int_c^d\big(K_1(s,ƭ)+K_2(s,ƭ)\big)\,ds\,.\,\frac{\theta\big(ḑ(\varkappa,u)\big)+\theta\big(ḑ(v,y)\big)}{2}$$

$$\le \frac{\theta\big(ḑ(\varkappa,u)\big)+\theta\big(ḑ(v,y)\big)}{2}.$$

Now, since θ is a non-decreasing function, we have

$$\theta\big(ḑ(\varkappa,u)\big)\le\theta\big(ḑ(\varkappa,u)+ḑ(v,y)\big)\quad\text{and}\quad\theta\big(ḑ(v,y)\big)\le\theta\big(ḑ(\varkappa,u)+ḑ(v,y)\big),$$

so that, we can obtain that $\frac{\theta(ḑ(\varkappa,u))+\theta(ḑ(v,y))}{2}\le\theta\big(ḑ(\varkappa,u)+ḑ(v,y)\big)=\psi\left(\frac{ḑ(\varkappa,u)+ḑ(v,y)}{2}\right)$, by using (3.5.3). Therefore, we get

$$\alpha\,\frac{ḑ\big(Ғ(\varkappa,y),Ғ(u,v)\big)+ḑ(Ғ(y,\varkappa),Ғ(v,u))}{2}\le\psi\left(\frac{ḑ(\varkappa,u)+ḑ(v,y)}{2}\right),$$

which is the contractive condition (3.3.1) for $\varphi(ƭ)=\alpha ƭ$, where $\alpha\in(0,1)$. Now, let $(\hat{\varkappa},\hat{y})\in Ӽ\times Ӽ$ be a coupled upper-lower solution of (3.5.1). Then, we have

$$\hat{\varkappa}(ƭ)\preccurlyeq\hat{y}(ƭ),$$

$$\hat{\varkappa}(ƭ)\le Ғ(\hat{\varkappa},\hat{y})(ƭ)\quad\text{and}\quad\hat{y}(ƭ)\ge Ғ(\hat{y},\hat{\varkappa})(ƭ),$$

for all ƭ ∈ I. Now, applying Theorems 3.3.1 and 3.3.2, Ғ has a unique coupled fixed point. Now, since $\hat{\varkappa}\preccurlyeq\hat{y}$, so that the hypotheses of Theorem 3.3.3 are satisfied and hence, there exists a unique $\varkappa\in Ӽ$ such that $\varkappa(ƭ)=Ғ(\varkappa,\varkappa)(ƭ)$ for all ƭ ∈ I. Therefore, the integral equation (3.5.1) has a unique solution.

Next, as an application of the results obtained in section 3.4, we now obtain the result for mappings with MMP satisfying a contractive condition of the integral type. Denote by Ʊ, the class of functions $\varpi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following conditions:

(i) ϖ is a Lebesgue – integrable function on each compact of $\mathbb{R}^+$;

(ii) for each $\varepsilon>0$, we have $\int_0^\varepsilon\varpi(ƭ)dƭ>0$.

Theorem 3.5.2. Let (Ӽ, ≼, ϸ) be a POCPMS and Ғ: Ӽ × Ӽ → Ӽ be a mapping with MMP on Ӽ. Suppose that, for all ϰ, y, u, v ∈ Ӽ with ϰ ≽ u and y ≼ v (or ϰ ≼ u and y ≽ v), we have

$$\int_0^{\frac{ϸ(Ғ(\varkappa,y),Ғ(u,v))+ϸ(Ғ(y,\varkappa),Ғ(v,u))}{2}}\varpi_1(ƭ)dƭ\le\int_0^{\frac{ϸ(\varkappa,u)+ϸ(y,v)}{2}}\varpi_1(ƭ)dƭ-\int_0^{\frac{ϸ(\varkappa,u)+ϸ(y,v)}{2}}\varpi_2(ƭ)dƭ,\tag{3.5.13}$$

where $\varpi_1,\varpi_2\in$ Ʊ. Suppose either

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ҳ has the property (P3), then Ғ has a coupled fixed point in Ҳ.

Proof. The functions $s \mapsto \int_0^s \varpi_i(ŧ)dŧ$ (for i = 1, 2) defined on $\mathbb{R}^+$ are in Φ and in Ψ. Now, the result follows immediately by Theorem 3.4.1.

FRAMEWORK OF CHAPTER - IV

In this chapter, we discuss coupled common fixed point results for some generalized and weak symmetric contraction conditions in POMS. The contractions involved in our results are extensions of Meir-Keeler contractions and (α, ψ) – contractions to the mappings having MgMP. Applications to solution of integral equations are also discussed. Further, a result of integral type is also established.

PUBLISHED WORK:

(1) Journal of Computational Analysis and Applications, 16 (3) (2014), pp. 438-454.

(2) Journal of Mathematics and Computer Science, 10 (1) (2014), pp. 23-46.

CHAPTER – IV

COUPLED FIXED POINTS UNDER SYMMETRIC CONTRACTIONS

Present chapter deals with some generalized and weak symmetric contractions in POMS. This chapter consists of five sections. Section 4.1 gives a brief introduction to some symmetric contractions. In section 4.2, we establish some coupled coincidence and coupled common fixed point results under the notion of generalized symmetric g-Meir-Keeler type contractions. Section 4.3 consists of coupled coincidence and coupled common fixed point results for mixed g-monotone mappings satisfying $(\alpha,\ \psi)$ – weak contractions. In section 4.4, as applications of the results proved in various sections of this chapter, the solutions of integral equations are discussed. In the last section 4.5, an application to the result of the integral type is also given.

Author's Original Contributions In This Chapter Are:

Theorems: 4.2.1, 4.2.2, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.3.5, 4.3.6, 4.4.1, 4.4.2, 4.5.1.

Lemma: 4.2.1.

Proposition: 4.2.1.

Definitions: 4.2.1, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.4.1, 4.4.2.

Corollaries: 4.2.1, 4.5.1.

Examples: 4.3.1, 4.3.2, 4.3.3.

Remarks: 4.2.1, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.5.1.

Assumptions: 4.4.1, 4.4.2.

4.1 INTRODUCTION

Generalizing and extending BCP in different ways has always been an area of great interest for researchers. In 1969, Meir-Keeler [38] generalized BCP by proving Theorem 2.1.1. Later on, Harjani et al. [151] proved a result which was a version of the Theorem 2.1.1 for continuous, non-decreasing self mappings in POMS. Recently, Samet [152] extended the work of Meir-Keeler [38] for the mappings with the mixed strict monotone property. In fact, Samet [152] defined the notion of generalized Meir-Keeler type function and using this notion, proved some coupled fixed point theorems in the setup of POCMS.

Recall that “for a partial ordering $\preccurlyeq$ on the non-empty set $\mathfrak{X}$, the strict order $\prec$ on $\mathfrak{X}$ is defined as $\varkappa \prec y$ means that $\varkappa \preccurlyeq y$ but $\varkappa \neq y$ for $\varkappa$, y in $\mathfrak{X}$”.

Definition 4.1.1 ([152]). Let $(\mathfrak{X}, \preccurlyeq)$ be a poset. The mapping $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is said to have **mixed strict monotone property**, if $F(\varkappa, y)$ is strictly increasing in $\varkappa$ and strictly decreasing in y.

For brevity, we write mixed strict monotone property as **MSMP**.

Definition 4.1.2 ([152]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be the given mapping. Then, F is said to be **generalized Meir-Keeler type function**, if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\varkappa \succcurlyeq u$, $y \preccurlyeq v$, we have

$$\varepsilon \leq \frac{1}{2}[d(\varkappa, u) + d(y, v)] < \varepsilon + \delta(\varepsilon) \ \text{ implies } \ d(F(\varkappa, y), F(u, v)) < \varepsilon. \quad (4.1.1)$$

Subsequently, Gordji et al. [153] gave the notion of mixed strict g-monotone property and extended the results of Bhaskar and Lakshmikantham [55] and Samet [152] under generalized g-Meir-Keeler type contractions.

Definition 4.1.3 ([153]). Let $(\mathfrak{X}, \preccurlyeq)$ be a poset and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings. Then F is said to have the **mixed strict g-monotone property**, if $F(\varkappa, y)$ is strictly g-increasing in $\varkappa$ and strictly g-decreasing in y.

In short, we call mixed strict g-monotone property as **MSgMP**.

Definition 4.1.4 ([153]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g: \mathfrak{X} \to \mathfrak{X}$ be two given mappings. Then, F is said to be **generalized g-Meir-Keeler type contraction**, if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\varkappa$, y, u, v in $\mathfrak{X}$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$,

$$\varepsilon \leq \frac{1}{2}[d(g\varkappa, gu) + d(gy, gv)] < \varepsilon + \delta(\varepsilon) \ \text{ implies } \ d(F(\varkappa, y), F(u, v)) < \varepsilon. \quad (4.1.2)$$

On the other hand, Abdeljawad et al. [154] proved some interesting coupled fixed point results in partially ordered partial metric space (POPMS) and remarked that the metrical analogue of their work which was obtained by Gordji et al. [153] has gaps. In fact, in [154] it was remarked that some of the results proved in [153] are not true if the partial ordering is obtained via non-strongly minihedral cones. By the same time, Berinde and Pacurar [155] introduced the notion of generalized symmetric Meir-Keeler contractions and complemented the results of Samet [152] by proving the following result:

Theorem 4.1.1 ([155]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a continuous mapping with MMP and is also a generalized symmetric Meir-Keeler mapping, that is, for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\varkappa$, y, u, v in $\mathfrak{X}$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$,

$$\varepsilon \leq \frac{1}{2}[d(\varkappa, u) + d(y, v)] < \varepsilon + \delta(\varepsilon)$$

implies $$\frac{1}{2}[d(F(\varkappa, y), F(u, v)) + d(F(y, \varkappa), F(v, u))] < \varepsilon. \qquad (4.1.3)$$

If $\mathfrak{X}$ has the property **(P3)** which states: "there exist two elements $\varkappa_0, y_0 \in \mathfrak{X}$ with either $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(y_0, \varkappa_0)$, or $\varkappa_0 \succcurlyeq F(\varkappa_0, y_0)$ and $y_0 \preccurlyeq F(y_0, \varkappa_0)$", then F has a coupled fixed point in $\mathfrak{X}$.

It was also illustrated in [155] that the contractive condition (4.1.3) is weaker than (4.1.2).

In 2012, in order to generalize BCP, Samet et al. [156] introduced the notions of α- ψ-contractive and α-admissible mappings and used these notions to establish the existence of fixed points in complete metric spaces.

Definition 4.1.5 ([156]). Let ($\mathfrak{X}$, d) be a metric space and T: $\mathfrak{X} \to \mathfrak{X}$ be a given mapping. Then, T is said to be an ***α- ψ*-contractive mapping**, if there exist functions $\alpha: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that

$$\alpha(\varkappa, y) d(T\varkappa, Ty) \leq \psi(d(\varkappa, y)), \qquad \text{for all } \varkappa, y \in \mathfrak{X}. \qquad (4.1.4)$$

Definition 4.1.6 ([156]). Let T: $\mathfrak{X} \to \mathfrak{X}$ and $\alpha: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$. The mapping T is called ***α*-admissible** if

$$\alpha(\varkappa, y) \geq 1 \implies \alpha(T\varkappa, Ty) \geq 1, \qquad \text{for } \varkappa, y \in \mathfrak{X}. \qquad (4.1.5)$$

Successively, Mursaleen et al. [157] defined (α, ψ) - contractive mappings and extended the notion of α - admissible mappings to establish some coupled fixed point results in POMS.

Definition 4.1.7 ([157]). Let $F: \mathfrak{X} \times \mathfrak{X} \to X$ and $\alpha: \mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$ be two mappings. Then F is said to be **(*α*) - admissible** if for all $\varkappa$, y, u, v $\in \mathfrak{X}$, we have

$$\alpha((\varkappa, y), (u, v)) \geq 1 \implies \alpha\left((F(\varkappa, y), F(y, \varkappa)), (F(u, v), F(v, u))\right) \geq 1. \qquad (4.1.6)$$

Definition 4.1.8 ([157]). Let ($\mathfrak{X}$, $\preccurlyeq$, d) be a POMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a given mapping. Then, F is said to be **(*α*, *ψ*) - contractive mapping** if there exist functions $\alpha: \mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that for all $\varkappa$, y, u, v $\in \mathfrak{X}$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$,

$$\alpha((\varkappa, y), (u, v))\, d(F(\varkappa, y), F(u, v)) \leq \psi\left(\frac{d(\varkappa, u) + d(y, v)}{2}\right). \qquad (4.1.7)$$

Theorem 4.1.2 ([157]). Let ($\mathfrak{X}$, $\preccurlyeq$, d) be a POCMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a continuous mapping with MMP on $\mathfrak{X}$. Suppose there exist two functions

α: $\mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that Ғ is (α, ψ)-contractive mapping (that is, (4.1.7) holds). Also, suppose that Ғ is (α) – admissible and $\mathfrak{X}$ has the following property:

(P4) "there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $\alpha\big((\varkappa_0, y_0), (Ғ(\varkappa_0, y_0), Ғ(y_0, \varkappa_0))\big) \geq 1$ and $\alpha\big((y_0, \varkappa_0), (Ғ(y_0, \varkappa_0), Ғ(\varkappa_0, y_0))\big) \geq 1$".

If $\mathfrak{X}$ has the property:

(P1) "there exist two elements $\varkappa_0$, $y_0 \in \mathfrak{X}$ with $\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ғ(y_0, \varkappa_0)$", then Ғ has a coupled fixed point in $\mathfrak{X}$.

Karapinar and Agarwal [158] considered a more general contractive condition and weakened the contraction (4.1.7). The main result in [158] is as follows:

Theorem 4.1.3 ([158]). Let ($\mathfrak{X}$, $\preccurlyeq$, ɖ) be a POCMS and Ғ: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a mapping with MMP on $\mathfrak{X}$. Suppose there exist $\psi \in$ CCF-Ψ and α: $\mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$ such that for all $\varkappa$, y, u, v $\in \mathfrak{X}$ with $\varkappa \succcurlyeq$ u and y $\preccurlyeq$ v,

$$\alpha\big((\varkappa, y), (u,v)\big)\left(\frac{ɖ\big(Ғ(\varkappa,y), Ғ(u,v)\big) + ɖ\big(Ғ(y,\varkappa), Ғ(v,u)\big)}{2}\right) \leq \psi\left(\frac{ɖ(\varkappa,u) + ɖ(y,v)}{2}\right). \qquad (4.1.8)$$

Also, suppose that Ғ is (α) – admissible and continuous and $\mathfrak{X}$ has the property (P4). If $\mathfrak{X}$ has the property (P1), then Ғ has a coupled fixed point in $\mathfrak{X}$.

It has also been shown respectively in [157] and [158], that one can still obtain the coupled fixed point for the mapping Ғ, if the continuity hypothesis of the mapping Ғ in Theorems 4.1.2 and 4.1.3 can be replaced by the following condition:

Assumption 4.1.1 ([157, 158]). $\mathfrak{X}$ has the property:

"If $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in $\mathfrak{X}$ such that $\alpha\big((\varkappa_n, y_n), (\varkappa_{n+1}, y_{n+1})\big) \geq 1$ and $\alpha\big((y_n, \varkappa_n), (y_{n+1}, \varkappa_{n+1})\big) \geq 1$ for all n, and $\lim_{n\to\infty} \varkappa_n = \varkappa$ and $\lim_{n\to\infty} y_n = y$, then $\alpha\big((\varkappa_n, y_n), (\varkappa, y)\big) \geq 1$ and $\alpha\big((y_n, \varkappa_n), (y, \varkappa)\big) \geq 1$ for all n".

4.2 COUPLED COMMON FIXED POINTS FOR GENERALIZED SYMMETRIC CONTRACTION

In this section, we introduce the notion of generalized symmetric g-Meir-Keeler type contraction and utilize it to establish some results for mappings with MSgMP in POMS. Our notion extends the notion of generalized symmetric Meir-Keeler contraction due to Berinde and Pacurar [155].

We now introduce our notion as follows:

Definition 4.2.1. Let $(\mathcal{X}, \preccurlyeq, d)$ be a POMS and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $g: \mathcal{X} \to \mathcal{X}$ be the two given mappings. We say that F is a **generalized symmetric g-Meir-Keeler type contraction** if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all $\varkappa$, y, u, v $\in \mathcal{X}$ with $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$ (or $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$),

$$\varepsilon \le \frac{1}{2}[d(g\varkappa, gu) + d(gy, gv)] < \varepsilon + \delta(\varepsilon),$$

implies $$\frac{1}{2}\left[d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big)\right] < \varepsilon. \qquad (4.2.1)$$

Definition 4.2.1 extends the notion of generalized symmetric Meir-Keeler type contraction (4.1.3) for a pair of mappings.

Proposition 4.2.1. Let $(\mathcal{X}, \preccurlyeq, d)$ be a POMS and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a given mapping. Assume that there exists some k, $0 < k < 1$ such that for all $\varkappa$, y, u, v in $\mathcal{X}$ with $\varkappa \succcurlyeq u$, $y \preccurlyeq v$, we have

$$d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big) \le k[d(\varkappa, u) + d(y, v)], \qquad (4.2.2)$$

then, F is a generalized symmetric Meir-Keeler type contraction.

Proof. Suppose that (4.2.2) holds for some k, $0 < k < 1$. Then, for all $\varepsilon > 0$, it is easy to check that (4.1.3) is satisfied with $\delta(\varepsilon) = \big((1/k) - 1\big)\varepsilon$.

Lemma 4.2.1. Let $(\mathcal{X}, \preccurlyeq, d)$ be a POMS and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $g: \mathcal{X} \to \mathcal{X}$ be the two given mappings. If F is a generalized symmetric g-Meir-Keeler type contraction, then, for all $\varkappa$, y, u, v in $\mathcal{X}$ with $g\varkappa \prec gu$, $gy \succcurlyeq gv$ (or $g\varkappa \preccurlyeq gu$, $gy \succ gv$) we have

$$d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big) < d(g\varkappa, gu) + d(gy, gv). \qquad (4.2.3)$$

Proof. W.L.O.G., suppose that $g\varkappa \prec gu$, $gy \succcurlyeq gv$ for $\varkappa$, y, u, v $\in \mathcal{X}$, then we have $d(g\varkappa, gu) + d(gy, gv) > 0$. Since F is generalized symmetric g-Meir-Keeler type contraction, for $\varepsilon = (1/2)[d(g\varkappa, gu) + d(gy, gv)]$, there exists a $\delta(\varepsilon) > 0$ such that, for all $\varkappa_0, y_0, u_0, v_0 \in \mathcal{X}$ with $g\varkappa_0 \prec gu_0$ and $gy_0 \succcurlyeq gv_0$,

$$\varepsilon \le \frac{1}{2}[d(g\varkappa_0, gu_0) + d(gy_0, gv_0)] < \varepsilon + \delta(\varepsilon),$$

implies $$\frac{1}{2}\left[d\big(F(\varkappa_0, y_0), F(u_0, v_0)\big) + d\big(F(y_0, \varkappa_0), F(v_0, u_0)\big)\right] < \varepsilon.$$

Then, the result follows by considering $\varkappa = \varkappa_0$, $y = y_0$, $u = u_0$, $v = v_0$.

We now establish our results as follows:

Theorem 4.2.1. Let $(\mathcal{X}, \preccurlyeq, d)$ be a POMS with the following properties:

(i) if $\{\varkappa_n\} \to \varkappa \in \mathcal{X}$ and $\varkappa_{n+1} \succ \varkappa_n$ for all $n \in \mathbb{N}$, then $\varkappa_n \prec \varkappa$ for all $n \in \mathbb{N}$;

(ii) if $\{y_n\} \to y \in \mathcal{X}$ and $y_{n+1} \prec y_n$ for all $n \in \mathbb{N}$, then $y_n \succ y$ for all $n \in \mathbb{N}$.

Let $\text{Ꞙ}: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings such that $g(\mathfrak{X})$ is a complete subspace of $\mathfrak{X}$ and $\text{Ꞙ}(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$. Also, suppose that

(a) Ꞙ has the MSgMP;

(b) Ꞙ is a generalized symmetric g-Meir-Keeler type contraction;

(c) $\mathfrak{X}$ has anyone of the following property:

(P5) "there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $g\varkappa_0 \prec \text{Ꞙ}(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq \text{Ꞙ}(y_0, \varkappa_0)$ (or, $g\varkappa_0 \preccurlyeq \text{Ꞙ}(\varkappa_0, y_0)$ and $gy_0 \succ \text{Ꞙ}(y_0, \varkappa_0)$)";

or

(P6) "there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $g\varkappa_0 \succ \text{Ꞙ}(\varkappa_0, y_0)$ and $gy_0 \preccurlyeq \text{Ꞙ}(y_0, \varkappa_0)$ (or, $g\varkappa_0 \succcurlyeq \text{Ꞙ}(\varkappa_0, y_0)$ and $gy_0 \prec \text{Ꞙ}(y_0, \varkappa_0)$)".

Then, Ꞙ and g have a coupled coincidence point in $\mathfrak{X}$.

Proof. W.L.O.G., suppose that there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $g\varkappa_0 \prec \text{Ꞙ}(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq \text{Ꞙ}(y_0, \varkappa_0)$. Since $\text{Ꞙ}(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$, choose $\varkappa_1$, y_1 in $\mathfrak{X}$ such that $g\varkappa_1 = \text{Ꞙ}(\varkappa_0, y_0)$, $gy_1 = \text{Ꞙ}(y_0, \varkappa_0)$. Again, we can choose $\varkappa_2$, y_2 in $\mathfrak{X}$ such that $g\varkappa_2 = \text{Ꞙ}(\varkappa_1, y_1)$, $gy_2 = \text{Ꞙ}(y_1, \varkappa_1)$.

Continuing this process, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in $\mathfrak{X}$ such that

$$g\varkappa_{n+1} = \text{Ꞙ}(\varkappa_n, y_n),\ gy_{n+1} = \text{Ꞙ}(y_n, \varkappa_n), \text{ for all } n \geq 0. \tag{4.2.4}$$

Using the conditions (a), (c) and mathematical induction, for all $n \geq 0$, we can obtain

$$g\varkappa_n \prec g\varkappa_{n+1} \tag{4.2.5}$$

and

$$gy_{n+1} \prec gy_n. \tag{4.2.6}$$

Denote $\varrho_n = \mathfrak{d}(g\varkappa_n, g\varkappa_{n+1}) + \mathfrak{d}(gy_n, gy_{n+1})$. (4.2.7)

Now, using (4.2.4), Lemma 4.2.1 and condition (b), we have

$$\begin{aligned}\varrho_n &= \mathfrak{d}(g\varkappa_n, g\varkappa_{n+1}) + \mathfrak{d}(gy_n, gy_{n+1})\\ &= \mathfrak{d}\big(\text{Ꞙ}(\varkappa_{n-1}, y_{n-1}), \text{Ꞙ}(\varkappa_n, y_n)\big) + \mathfrak{d}(\text{Ꞙ}(y_{n-1}, \varkappa_{n-1}), \text{Ꞙ}(y_n, \varkappa_n))\\ &< \mathfrak{d}(g\varkappa_{n-1}, g\varkappa_n) + \mathfrak{d}(gy_{n-1}, gy_n) = \varrho_{n-1}.\end{aligned} \tag{4.2.8}$$

Therefore, $\{\varrho_n\}$ is a decreasing sequence, so there exists some $\varrho^* \geq 0$ such that $\lim_{n\to\infty} \varrho_n = \varrho^*$. We claim that $\varrho^* = 0$. On the contrary, assume that $\varrho^* \neq 0$. Then there exists some $m \in \mathbb{N}$ such that, for any $n \geq m$, we have

$$\varepsilon \leq \varrho_n/2 = \frac{1}{2}[\mathfrak{d}(g\varkappa_n, g\varkappa_{n+1}) + \mathfrak{d}(gy_n, gy_{n+1})] < \varepsilon + \delta(\varepsilon), \tag{4.2.9}$$

where $\varepsilon = \varrho^*/2$ and $\delta(\varepsilon)$ is chosen by condition (b). In particular, for $n = m$, we have

$$\varepsilon \leq (\varrho_m/2) = \frac{1}{2}[\mathfrak{d}(g\varkappa_m, g\varkappa_{m+1}) + \mathfrak{d}(gy_m, gy_{m+1})] < \varepsilon + \delta(\varepsilon). \tag{4.2.10}$$

Then, using condition (b), it follows that

$$\frac{1}{2}\left[d\big(F(\varkappa_m, y_m), F(\varkappa_{m+1}, y_{m+1})\big) + d\big(F(y_m, \varkappa_m), F(y_{m+1}, \varkappa_{m+1})\big)\right] < \varepsilon, \quad (4.2.11)$$

and hence, using (4.2.4), we get

$$\frac{1}{2}\left[d(g\varkappa_{m+1}, g\varkappa_{m+2}) + d(gy_{m+1}, gy_{m+2})\right] < \varepsilon, \quad (4.2.12)$$

a contradiction to (4.2.9) for n = m + 1. Therefore, we have $\varrho^* = 0$, so that

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} [d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})] = 0. \quad (4.2.13)$$

We now claim that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let $\varepsilon > 0$ be arbitrary. Then, by (4.2.13), there exists some $k \in \mathbb{N}$ such that

$$\frac{1}{2}\left[d\big(g\varkappa_k, g\varkappa_{k+1}\big) + d\big(gy_k, gy_{k+1}\big)\right] < \delta(\varepsilon). \quad (4.2.14)$$

W.L.O.G., suppose k be chosen so large that $\delta(\varepsilon) \leq \varepsilon$ and consider the set

$$P = \left\{ \begin{array}{r} (g\varkappa, gy): (\varkappa, y) \in X^2, d\big(g\varkappa, g\varkappa_k\big) + d\big(gy, gy_k\big) < 2\big(\varepsilon + \delta(\varepsilon)\big), \\ \text{and } g\varkappa \succ g\varkappa_k, gy \preccurlyeq gy_k \end{array} \right\}. \quad (4.2.15)$$

We show that

$$(g\varkappa, gy) \in P \text{ implies that } (F(\varkappa, y), F(y, \varkappa)) \in P, \text{ where } \varkappa, y \in X. \quad (4.2.16)$$

Let $(g\varkappa, gy) \in P$. Then, by triangle inequality and (4.2.14), we get

$$\begin{aligned}
&\frac{1}{2}\left[d\Big(g\varkappa_k, F(\varkappa, y)\Big) + d\Big(gy_k, F(y, \varkappa)\Big)\right] \\
&\quad \leq \frac{1}{2}\left[d\big(g\varkappa_k, g\varkappa_{k+1}\big) + d\Big(g\varkappa_{k+1}, F(\varkappa, y)\Big)\right] + \frac{1}{2}\left[d\big(gy_k, gy_{k+1}\big) + d\Big(gy_{k+1}, F(y, \varkappa)\Big)\right] \\
&\quad = \frac{1}{2}\left[d\big(g\varkappa_k, g\varkappa_{k+1}\big) + d\big(gy_k, gy_{k+1}\big)\right] + \frac{1}{2}\left[d\Big(g\varkappa_{k+1}, F(\varkappa, y)\Big) + d\Big(gy_{k+1}, F(y, \varkappa)\Big)\right] \\
&\quad < \delta(\varepsilon) + \frac{1}{2}\begin{bmatrix} d\Big(F(\varkappa, y), F\big(\varkappa_k, y_k\big)\Big) \\ +d\big(F(y, \varkappa), F(y_k, \varkappa_k)\big) \end{bmatrix}.
\end{aligned} \quad (4.2.17)$$

We consider the following two cases:

Case 1. $(1/2)\big[d\big(g\varkappa, g\varkappa_k\big) + d\big(gy, gy_k\big)\big] \leq \varepsilon$. Then, using Lemma 4.2.1 and the Definition of P, the inequality (4.2.17) becomes

$$\begin{aligned}
\frac{1}{2}\left[d\Big(g\varkappa_k, F(\varkappa, y)\Big) + d\Big(gy_k, F(y, \varkappa)\Big)\right] &< \delta(\varepsilon) + \frac{1}{2}\begin{bmatrix} d\Big(F(\varkappa, y), F\big(\varkappa_k, y_k\big)\Big) \\ + d\big(F(y, \varkappa), F(y_k, \varkappa_k)\big) \end{bmatrix} \\
&< \delta(\varepsilon) + \frac{1}{2}\left[d\big(g\varkappa, g\varkappa_k\big) + d\big(gy, gy_k\big)\right] \leq \delta(\varepsilon) + \varepsilon.
\end{aligned} \quad (4.2.18)$$

Case 2. $\varepsilon < (1/2)\big[d\big(g\varkappa, g\varkappa_k\big) + d\big(gy, gy_k\big)\big] < \delta(\varepsilon) + \varepsilon$.

In this case, we have

$$\varepsilon < (1/2)\big[d\big(g\varkappa, g\varkappa_k\big) + d\big(gy, gy_k\big)\big] < \delta(\varepsilon) + \varepsilon. \quad (4.2.19)$$

Then, since $g\varkappa \succ g\varkappa_k$ and $gy \preccurlyeq gy_k$, using the condition (b), we obtain

$$\frac{1}{2}\left[d\left(F(\varkappa,y),F(\varkappa_k,y_k)\right)+d\left(F(y,\varkappa),F(y_k,\varkappa_k)\right)\right]<\varepsilon. \tag{4.2.20}$$

Using (4.2.20) in (4.2.17), we obtain

$$\frac{1}{2}\left[d\left(g\varkappa_k,F(\varkappa,y)\right)+d\left(gy_k,F(y,\varkappa)\right)\right]<\delta(\varepsilon)+\varepsilon. \tag{4.2.21}$$

Since F satisfies the MSgMP and $(g\varkappa, gy) \in P$, it follows that

$$F(\varkappa, y) \succ g\varkappa_k \text{ and } F(y,\varkappa) \prec gy_k. \tag{4.2.22}$$

Also, since $F(X \times X) \subseteq g(X)$ we have $(F(\varkappa, y), F(y, \varkappa)) \in P$, that is (4.2.16) holds. Using (4.2.14), we get $(g\varkappa_{k+1}, gy_{k+1}) \in P$. Then, using (4.2.16), we obtain

$$\begin{aligned}&\left(g\varkappa_{k+1},gy_{k+1}\right)\in P\\ \Longrightarrow&\left(F(\varkappa_{k+1},y_{k+1}),F(y_{k+1},\varkappa_{k+1})\right)=\left(g\varkappa_{k+2},gy_{k+2}\right)\in P\\ \Longrightarrow&\left(F(\varkappa_{k+2},y_{k+2}),F(y_{k+2},\varkappa_{k+2})\right)=\left(g\varkappa_{k+3},gy_{k+3}\right)\in P\\ &\Longrightarrow\cdots\Longrightarrow(g\varkappa_n,gy_n)\in P\Longrightarrow\cdots.\end{aligned} \tag{4.2.23}$$

Then, for all $n > k$, we have $(g\varkappa_n, gy_n) \in P$. This implies, for all $n, m > k$, that

$$\begin{aligned}&d(g\varkappa_n,g\varkappa_m)+d(gy_n,gy_m)\\ &\quad\le d(g\varkappa_n,g\varkappa_k)+d(g\varkappa_k,g\varkappa_m)+d(gy_n,gy_k)+d(gy_k,gy_m)\\ &\quad=\left[d(g\varkappa_n,g\varkappa_k)+d(gy_n,gy_k)\right]+\left[d(g\varkappa_k,g\varkappa_m)+d(gy_k,gy_m)\right]\\ &\quad\le 4\left(\varepsilon+\delta(\varepsilon)\right)\le 8\varepsilon.\end{aligned}$$

Hence, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences, then, by completeness of $g(X)$ there exist $\varkappa, y \in X$ such that

$$\lim_{n\to\infty} d(g\varkappa_n, g\varkappa)=0 \quad \text{and} \quad \lim_{n\to\infty} d(gy_n, gy)=0. \tag{4.2.24}$$

Since $\{g\varkappa_n\}$ and $\{gy_n\}$ are monotone increasing and decreasing sequences, respectively, then by conditions (i) and (ii), we get

$$g\varkappa_n \prec g\varkappa \quad \text{and} \quad gy_n \succ gy, \tag{4.2.25}$$

for each $n \geq 0$. Then, on using (4.2.25) and Lemma 4.2.1, along with the condition (b), we get

$$\begin{aligned}&d\left(g\varkappa_{n+1},F(\varkappa,y)\right)+d\left(gy_{n+1},F(y,\varkappa)\right)\\ &\quad=d\left(F(\varkappa_n,y_n),F(\varkappa,y)\right)+d\left(F(y_n,\varkappa_n),F(y,\varkappa)\right)\\ &\quad<d(g\varkappa_n,g\varkappa)+d(gy_n,gy).\end{aligned} \tag{4.2.26}$$

Taking $n \to \infty$ in (4.2.26) and using (4.2.24), we get

$$d\left(g\varkappa,F(\varkappa,y)\right)+d\left(gy,F(y,\varkappa)\right)\le\lim_{n\to\infty}\left[d(g\varkappa_n,g\varkappa)+d(gy_n,gy)\right], \tag{4.2.27}$$

which gives us $F(\varkappa, y) = g\varkappa$, $F(y, \varkappa) = gy$. Hence, we have proved our result.

Corollary 4.2.1. Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POCMS with properties (i) and (ii) of Theorem 4.2.1. Let $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a mapping. Also, suppose that

(d) F has the MSMP;

(e) F is generalized symmetric Meir-Keeler type contraction;

(f) $\mathfrak{X}$ has the following property:

(P7) "there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $\varkappa_0 \prec F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(y_0, \varkappa_0)$ (or, $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succ F(y_0, \varkappa_0)$)";

or

(P8) "there exist $\varkappa_0, y_0 \in \mathfrak{X}$ such that $\varkappa_0 \succ F(\varkappa_0, y_0)$ and $y_0 \preccurlyeq F(y_0, \varkappa_0)$ (or $\varkappa_0 \succcurlyeq F(\varkappa_0, y_0)$ and $y_0 \prec F(y_0, \varkappa_0)$)".

Then, F has a coupled fixed point in $\mathfrak{X}$.

Proof. Taking g to be the identity mapping on $\mathfrak{X}$ in Theorem 4.2.1, the result follows immediately.

Remark 4.2.1. Corollary 4.2.1 improves Theorem 4.1.1 (Berinde and Pacurar [155]).

Coupled Common Fixed Points

Now, we establish the existence and uniqueness of the coupled common fixed point under the hypotheses of Theorem 4.2.1 and an additional hypothesis. But first, we need to consider the following notion:

For a poset $(\mathfrak{X}, \preccurlyeq)$, endow $\mathfrak{X} \times \mathfrak{X}$ with the following order $\preccurlyeq_g$:

"$(u, v) \preccurlyeq_g (\varkappa, y)$ iff $gu \prec g\varkappa$ and $gy \preccurlyeq gv$ for all $(\varkappa, y), (u, v) \in \mathfrak{X} \times \mathfrak{X}$". (4.2.28)

Here, we say that (u, v) and $(\varkappa, y)$ are g-comparable if either

$$(u, v) \preccurlyeq_g (\varkappa, y) \quad \text{or} \quad (\varkappa, y) \preccurlyeq_g (u, v).$$

If g is identity on $\mathfrak{X}$, then we say that (u, v) and $(\varkappa, y)$ are comparable and denote this fact by: $(u, v) \preccurlyeq (\varkappa, y)$.

Theorem 4.2.2. In addition to the hypotheses of Theorem 4.2.1 suppose that "for all non g-comparable points $(\varkappa, y), (\varkappa^*, y^*) \in \mathfrak{X} \times \mathfrak{X}$, there exists a point $(a, b) \in \mathfrak{X} \times \mathfrak{X}$ such that $(F(a, b), F(b, a))$ is comparable to both $(g\varkappa, gy)$ and $(g\varkappa^*, gy^*)$". Further, let the pair (F, g) be compatible. Then, F and g have a unique coupled common fixed point in $\mathfrak{X}$.

Proof. By Theorem 4.2.1 the set of coupled coincidence points of F and g is non-empty. We first prove that, if $(\varkappa, y)$ and $(\varkappa^*, y^*)$ are coupled coincidence points of F and g, then

$$g\varkappa = g\varkappa^* \quad \text{and} \quad gy = gy^*. \tag{4.2.29}$$

We distinguish the following cases:

Case 1. $(\varkappa, y)$ is g-comparable to $(\varkappa^*, y^*)$ w.r.t. the ordering in $X \times X$, where

$$F(\varkappa, y) = g\varkappa,\ F(y, \varkappa) = gy,\ F(\varkappa^*, y^*) = g\varkappa^*,\ F(y^*, \varkappa^*) = gy^*. \tag{4.2.30}$$

W.L.O.G., suppose that

$$g\varkappa = F(\varkappa, y) \prec F(\varkappa^*, y^*) = g\varkappa^*,\ gy = F(y, \varkappa) \succcurlyeq F(y^*, \varkappa^*) = gy^*. \tag{4.2.31}$$

Using Lemma 4.2.1, we get

$$\begin{aligned}0 &< d(g\varkappa, g\varkappa^*) + d(gy^*, gy)\\ &= d(F(\varkappa, y), F(\varkappa^*, y^*)) + d(F(y, \varkappa), F(y^*, \varkappa^*))\\ &< d(g\varkappa, g\varkappa^*) + d(gy^*, gy), \text{ a contradiction.}\end{aligned}$$

Hence, we have $(g\varkappa, gy) = (g\varkappa^*, gy^*)$. Therefore, (4.2.29) holds.

Case 2. $(\varkappa, y)$ is not g-comparable to $(\varkappa^*, y^*)$.

By assumption, there exists some $(a, b) \in X \times X$ such that $(F(a, b), F(b, a))$ is comparable to both $(g\varkappa, gy)$ and $(g\varkappa^*, gy^*)$. Then, we have

$$g\varkappa = F(\varkappa, y) \prec F(a, b),\ F(\varkappa^*, y^*) = g\varkappa^* \prec F(a, b), \tag{4.2.32}$$

and $$gy = F(y, \varkappa) \succcurlyeq F(b, a),\ F(y^*, \varkappa^*) = gy^* \succcurlyeq F(b, a). \tag{4.2.33}$$

Setting $\varkappa = \varkappa_0$, $y = y_0$, $a = a_0$, $b = b_0$ and $\varkappa^* = \varkappa_0^*$, $y^* = y_0^*$ as in the proof of Theorem 4.2.1, for $n \geq 0$, we can obtain

$$\begin{aligned}g\varkappa_{n+1} &= F(\varkappa_n, y_n), \qquad gy_{n+1} = F(y_n, \varkappa_n),\\ ga_{n+1} &= F(a_n, b_n), \qquad gb_{n+1} = F(b_n, a_n),\\ g\varkappa_{n+1}^* &= F(\varkappa_n^*, y_n^*), \qquad gy_{n+1}^* = F(y_n^*, \varkappa_n^*).\end{aligned} \tag{4.2.34}$$

Since $(F(\varkappa, y), F(y, \varkappa)) = (g\varkappa, gy) = (g\varkappa_1, gy_1)$ is comparable with $(F(a, b), F(b, a)) = (ga_1, gb_1)$, we get $g\varkappa \prec ga_1$ and $gy \succcurlyeq gb_1$. Using the fact that F has the MSgMP, we have $g\varkappa \prec ga_n$ and $gb_n \prec gy$ for all $n \geq 2$. Then, using Lemma 4.2.1, we get

$$\begin{aligned}0 &< d(g\varkappa, ga_{n+1}) + d(gy, gb_{n+1})\\ &= d\big(F(\varkappa, y), F(a_n, b_n)\big) + d\big(F(y, \varkappa), F(b_n, a_n)\big)\\ &< d(g\varkappa, ga_n) + d(gy, gb_n).\end{aligned} \tag{4.2.35}$$

Denote $d_n = d(g\varkappa, ga_n) + d(gy, gb_n)$, then, using (4.2.35), it follows that $\{d_n\}$ is a decreasing sequence and hence, converges to some $d \geq 0$. We claim that $d = 0$. On the contrary, assume that $d > 0$. Then, there exists some $p \in \mathbb{N}$ such that for $n \geq p$, we have

$$\varepsilon \leq \frac{d_n}{2} = \frac{1}{2}[d(g\varkappa, ga_n) + d(gy, gb_n)] < \varepsilon + \delta(\varepsilon), \tag{4.2.36}$$

where, $\varepsilon = \frac{d}{2}$ and $\delta(\varepsilon)$ is chosen by condition (b) of Theorem 4.2.1.

In particular, for n = p, we have

$$\varepsilon \leq \frac{d_p}{2} = \frac{1}{2}\left[d\big(g\varkappa, ga_p\big) + d\big(gy, gb_p\big)\right] < \varepsilon + \delta(\varepsilon). \tag{4.2.37}$$

Then, using the condition (b) of Theorem 4.2.1, we get

$$\frac{1}{2}\left[d\left(F(\varkappa, y), F\big(a_p, b_p\big)\right) + d\left(F(y, \varkappa), F\big(b_p, a_p\big)\right)\right] < \varepsilon, \tag{4.2.38}$$

that is, $\frac{1}{2}\left[d\big(g\varkappa, ga_{p+1}\big) + d\big(gy, gb_{p+1}\big)\right] < \varepsilon,$ (4.2.39)

which contradicts (4.2.36) for n = p + 1. Therefore $d = 0$, and hence

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty}[d(g\varkappa, ga_n) + d(gy, gb_n)] = 0. \tag{4.2.40}$$

Similarly, we can obtain

$$\lim_{n\to\infty}[d(g\varkappa^*, ga_n) + d(gy^*, gb_n)] = 0. \tag{4.2.41}$$

Now, using the triangle inequality, we get

$$\begin{aligned} &d(g\varkappa, g\varkappa^*) + d(gy, gy^*) \\ &\leq d(g\varkappa, ga_n) + d(ga_n, g\varkappa^*) + d(gy, gb_n) + d(gb_n, gy^*) \\ &= [d(g\varkappa, ga_n) + d(gy, gb_n)] + [d(g\varkappa^*, ga_n) + d(gy^*, gb_n)] \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.2.42}$$

Therefore, we can get $d(g\varkappa, g\varkappa^*) = 0$ and $d(gy, gy^*) = 0$. Thus, (4.2.29) holds in both the cases. Now, since $g\varkappa = F(\varkappa, y)$, $gy = F(y, \varkappa)$ and (F, g) is a compatible pair of mappings, then using Lemma 3.2.1, which states, "The pair of compatible mappings $F: X \times X \to X$ and $g: X \to X$ commutes at their coincidence points", we obtain that

$$gg\varkappa = gF(\varkappa, y) = F(g\varkappa, gy) \text{ and } ggy = gF(y, \varkappa) = F(gy, g\varkappa). \tag{4.2.43}$$

Denote $g\varkappa = z$, $gy = w$. Then, using (4.2.43), we get

$$gz = F(z, w) \text{ and } gw = F(w, z). \tag{4.2.44}$$

Thus, (z, w) is a coupled coincidence point.

Then, using (4.2.29) with $\varkappa^* = z$ and $y^* = w$, it follows that $gz = g\varkappa$ and $gw = gy$, that is,

$$gz = z, \; gw = w. \tag{4.2.45}$$

Now, using (4.2.44) and (4.2.45), we have $z = gz = F(z, w)$ and $w = gw = F(w, z)$. Thus, (z, w) is a coupled common fixed point of F and g. To show the uniqueness, suppose (s, l) be any coupled common fixed point of F and g. Then using (4.2.29), we can obtain $s = gs = gz = z$ and $l = gl = gw = w$. This completes the proof.

4.3 COUPLED COMMON FIXED POINTS FOR (α, ψ) – WEAK CONTRACTIONS

In this section, we define the notions of (α, ψ) – weak contractions in POMS for coupled fixed point problems. We utilize these notions to improve the recent results of Mursaleen et al. [157] and Karapinar and Agarwal [158] (that is, Theorems 4.1.2 and 4.1.3, respectively) and generalize the works of Bhaskar and Lakshmikantham [55], Berinde [149] and Jain et al. [159] (that is, Theorems 2.1.14, 3.1.1 and Corollary 3.2.1, respectively).

We first define the following notions:

Definition 4.3.1. Let (X, $\preccurlyeq$, d) be a POMS. The mapping $F: X \times X \to X$ is said to be **(α, ψ) – weak contraction** if there exist functions $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that

$$\alpha\big((x, y), (u,v)\big)\left(\frac{d\big(F(x,y), F(u,v)\big) + d\big(F(y,x), F(v,u)\big)}{2}\right) \leq \psi\left(\frac{d(x,u) + d(y,v)}{2}\right), \qquad (4.3.1)$$

for all x, y, u, v $\in X$ with $x \succcurlyeq$ u and y $\preccurlyeq$ v (or $x \preccurlyeq$ u and y $\succcurlyeq$ v).

Definition 4.3.2. Let (X, $\preccurlyeq$, d) be a POMS and $F: X \times X \to X$, $g: X \to X$ be two mappings. Then, F is said to be **(α, ψ) - weak contraction w.r.t. g**, if there exist two functions $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that

$$\alpha\big((gx, gy), (gu, gv)\big)\left(\frac{d\big(F(x,y), F(u,v)\big) + d\big(F(y,x), F(v,u)\big)}{2}\right) \leq \psi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right), \quad (4.3.2)$$

for all x, y, u, v $\in X$ with $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$ (or $gx \preccurlyeq gu$ and $gy \succcurlyeq gv$).

Definition 4.3.3. Let $F: X \times X \to X$, $g: X \to X$ and $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ be mappings. The mapping F is said to be **(α) - admissible w.r.t. g** if

$$\alpha\big((gx, gy), (gu, gv)\big) \geq 1 \Longrightarrow \alpha\Big(\big(F(x, y), F(y, x)\big), \big(F(u, v), F(v, u)\big)\Big) \geq 1, \quad (4.3.3)$$

for all x, y, u, v $\in X$.

On taking g to be the identity mapping in Definition 4.3.3, we get the definition of (α) - admissible mappings.

Now, we establish our results as follows:

Theorem 4.3.1. Let (X, $\preccurlyeq$, d) be a POCMS and $F: X \times X \to X$, $g: X \to X$ be two mappings such that F has the MgMP on X and g is continuous. Assume that there exist functions $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that F is (α, ψ) - weak contraction w.r.t. g. Also, suppose that

(i) F is (α) - admissible w.r.t. g;

(ii) there exist x_0, $y_0 \in X$ such that

$$\alpha\big((g\varkappa_0, gy_0), (F(\varkappa_0, y_0), F(y_0, \varkappa_0))\big) \geq 1;$$

(iii) $F(X \times X) \subseteq g(X)$;

(iv) (F, g) is compatible;

(v) F is continuous.

If in the hypothesis (ii), the elements $\varkappa_0$, $y_0 \in X$ be chosen so that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$, then, F and g have a coupled coincidence point in X.

Proof: By hypothesis, there exist $\varkappa_0$, y_0 in X be such that

$$\alpha\big((g\varkappa_0, gy_0), (F(\varkappa_0, y_0), F(y_0, \varkappa_0))\big) \geq 1,\ g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0) \text{ and } gy_0 \succcurlyeq F(y_0, \varkappa_0).$$

Since $F(X \times X) \subseteq g(X)$ and F has MgMP in X, then as in the proof of Theorem 3.2.1, sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in X such that

$$g\varkappa_{n+1} = F(\varkappa_n, y_n),\ gy_{n+1} = F(y_n, \varkappa_n), \text{ for all } n \geq 0, \tag{4.3.4}$$

and

$$g\varkappa_n \preccurlyeq g\varkappa_{n+1},\ gy_n \preccurlyeq gy_{n+1}, \text{ for all } n \geq 0. \tag{4.3.5}$$

We suppose either $g\varkappa_{n+1} = F(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = F(y_n, \varkappa_n) \neq gy_n$, otherwise the result is trivial.

Since F is (α) - admissible w.r.t. g, we have

$$\alpha\big((g\varkappa_0, gy_0), (g\varkappa_1, gy_1)\big) = \alpha\big((g\varkappa_0, gy_0), (F(\varkappa_0, y_0), F(y_0, \varkappa_0))\big) \geq 1$$

$$\Rightarrow \alpha\big((F(\varkappa_0, y_0), F(y_0, \varkappa_0)), (F(\varkappa_1, y_1), F(y_1, \varkappa_1))\big) = \alpha\big((g\varkappa_1, gy_1), (g\varkappa_2, gy_2)\big) \geq 1.$$

Then, inductively, for all $n \in \mathbb{N}$, we obtain

$$\alpha\big((g\varkappa_n, gy_n), (g\varkappa_{n+1}, gy_{n+1})\big) \geq 1. \tag{4.3.6}$$

Since F is (α, ψ) – weak contraction w.r.t. g, using (4.3.6), we get

$$\begin{aligned}
&\frac{d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})}{2} \\
&= \frac{d(F(\varkappa_{n-1}, y_{n-1}), F(\varkappa_n, y_n)) + d(F(y_{n-1}, \varkappa_{n-1}), F(y_n, \varkappa_n))}{2} \\
&\leq \alpha\big((g\varkappa_{n-1}, gy_{n-1}), (g\varkappa_n, gy_n)\big)\left(\frac{d(F(\varkappa_{n-1}, y_{n-1}), F(\varkappa_n, y_n)) + d(F(y_{n-1}, \varkappa_{n-1}), F(y_n, \varkappa_n))}{2}\right) \\
&\leq \psi\left(\frac{d(g\varkappa_{n-1}, g\varkappa_n) + d(gy_{n-1}, gy_n)}{2}\right).
\end{aligned} \tag{4.3.7}$$

Repeating the above process, we obtain

$$\frac{d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})}{2} \leq \psi^n\left(\frac{d(g\varkappa_0, g\varkappa_1) + d(gy_0, gy_1)}{2}\right), \text{ for all } n \in \mathbb{N}.$$

For $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\varepsilon)} \psi^n\left(\frac{d(g\varkappa_0, g\varkappa_1) + d(gy_0, gy_1)}{2}\right) < \varepsilon/2$.

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\varepsilon)$. Then, using the triangle inequality, we have

$$\begin{aligned}
&\frac{d(g\varkappa_n, g\varkappa_m) + d(gy_n, gy_m)}{2} \leq \sum_{k=n}^{m-1} \frac{d(g\varkappa_k, g\varkappa_{k+1}) + d(gy_k, gy_{k+1})}{2} \\
&\leq \sum_{k=n}^{m-1} \psi^k\left(\frac{d(g\varkappa_0, g\varkappa_1) + d(gy_0, gy_1)}{2}\right) \leq \sum_{n \geq n(\varepsilon)} \psi^n\left(\frac{d(g\varkappa_0, g\varkappa_1) + d(gy_0, gy_1)}{2}\right) < \varepsilon/2,
\end{aligned}$$

which implies that $d(g\varkappa_n, g\varkappa_m) + d(gy_n, gy_m) < \varepsilon$. Thus, it follows that $\{g(\varkappa_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X. Then, by the completeness of X, there exist $\varkappa$, y ∈ X such that

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_{n+1} = \varkappa,\ \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_{n+1} = y. \tag{4.3.8}$$

Since the pair (F, g) is compatible, we have

$$\begin{aligned} \lim_{n\to\infty} d\big(g(F(\varkappa_n, y_n)), F(g\varkappa_n, gy_n)\big) &= 0,\\ \lim_{n\to\infty} d\big(g(F(y_n, \varkappa_n)), F(gy_n, g\varkappa_n)\big) &= 0. \end{aligned} \tag{4.3.9}$$

Finally, we show that $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$.

Using the triangle inequality, for all n ≥ 0, we have

$$d\big(g\varkappa, F(g\varkappa_n, gy_n)\big) \le d\big(g\varkappa, g(F(\varkappa_n, y_n))\big) + d\big(g(F(\varkappa_n, y_n)), F(g\varkappa_n, gy_n)\big). \tag{4.3.10}$$

Letting n → ∞ in (4.3.10), then using the continuities of F, g and using (4.3.8), (4.3.9), we can obtain $d(g\varkappa, F(\varkappa, y)) = 0$ and hence, $g\varkappa = F(\varkappa, y)$. Similarly, $gy = F(y, \varkappa)$. Therefore, F and g have a coupled coincidence point in X.

Next, we give an example in support of Theorem 4.3.1.

Example 4.3.1. Consider the POCMS (X, ≼, d) with X = ℝ, the natural ordering ≤ of the real numbers as the partial ordering ≼ and $d(\varkappa, y) = |\varkappa - y|$ for all $\varkappa$, y ∈ X. Define the mapping $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ by

$$\alpha\big((\varkappa, y), (u, v)\big) = \begin{cases} 1, & \text{if } \varkappa \ge u,\ y \le v,\ \text{or } \varkappa \le u,\ y \ge v,\\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(f) = \frac{3}{5} f$, for $f \in \mathbb{R}^+$.

Define $F: X \times X \to X$ by $F(\varkappa, y) = \frac{2\varkappa - y}{10}$ for $\varkappa$, y ∈ X and g: X → X by $g\varkappa = \frac{\varkappa}{2}$ for $\varkappa \in X$. Then, F has the MgMP on X and the pair (F, g) is compatible. Clearly, F is (α) – admissible w.r.t. to g. Now, we show that F is (α, ψ) – weak contraction w.r.t. g. Now, if $\alpha\big((\varkappa, y), (u, v)\big) = 0$, then the result holds trivially. Suppose that $\alpha\big((\varkappa, y), (u, v)\big) = 1$. W.L.O.G., assume that $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$ so that $\varkappa \ge u$ and $y \le v$, then, we have

$$\alpha\big((\varkappa, y), (u, v)\big)\left(\frac{d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big)}{2}\right)$$

$$= \frac{d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big)}{2} = \frac{1}{20}|2(\varkappa - u) - (y - v)| + \frac{1}{20}|2(y - v) - (\varkappa - u)|$$

$$\le \frac{3}{5}\left(\frac{\left|\frac{\varkappa}{2} - \frac{u}{2}\right| + \left|\frac{y}{2} - \frac{v}{2}\right|}{2}\right) = \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right).$$

Further, choosing $\varkappa_0 = -1$ and $y_0 = 1$ in $\mathfrak{X}$ we have $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Hence, all the conditions of Theorem 4.3.1 are satisfied and the point (0, 0) is a coupled coincidence point of F and g.

Now, in order to relax the compatible hypothesis of the pair (F, g) and to replace the continuity assumption of F and g, we require the following notion:

Definition 4.3.4. Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POMS. Consider the function $\alpha: \mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$. We say that $(\mathfrak{X}, \preccurlyeq, d)$ is ***α*-regular**, if

(1) for each convergent sequences $\{\varkappa_n\}$ and $\{y_n\}$ in $\mathfrak{X}$ with $\alpha\big((\varkappa_n, y_n), (\varkappa_{n+1}, y_{n+1})\big) \geq 1$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varkappa_n = \varkappa \in \mathfrak{X}$ and $\lim_{n\to\infty} y_n = y \in \mathfrak{X}$, we have $\alpha\big((\varkappa_n, y_n), (x, y)\big) \geq 1$;

(2) the pairs $(\varkappa_n, y_n)$ and $(\varkappa, y)$ are comparable w.r.t. partial ordering in $\mathfrak{X} \times \mathfrak{X}$.

Theorem 4.3.2. Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POMS and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings such that F has the MgMP on $\mathfrak{X}$. Suppose there exist functions $\alpha: \mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that F is (α, ψ) – weak contraction w.r.t. g. Suppose that

(vi) hypotheses (i), (ii), (iii) of Theorem 4.3.1 hold and the range space $(g(\mathfrak{X}), d)$ is complete;

(vii) $(\mathfrak{X}, \preccurlyeq, d)$ is α - regular.

If in the hypothesis (ii), the elements $\varkappa_0, y_0 \in \mathfrak{X}$ be such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Then, F and g have a coupled coincidence point in $\mathfrak{X}$.

Proof. Following the proof of Theorem 4.3.1, we can obtain that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences in complete metric space $(g(\mathfrak{X}), d)$, so there exist $\varkappa$, y in $\mathfrak{X}$ such that

$$\lim_{n\to\infty} d(g\varkappa_n, g\varkappa) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gy_n, gy) = 0. \tag{4.3.11}$$

Also, using (4.3.6) and the hypothesis (vii), for all $n \in \mathbb{N}$, we get

$$\alpha\big((g\varkappa_n, gy_n), (g\varkappa, gy)\big) \geq 1, \tag{4.3.12}$$

and the pairs $(g\varkappa_n, g\varkappa)$ and (gy_n, gy) are comparable. We suppose $(g\varkappa_n, gy_n) \neq (g\varkappa, gy)$ for all n, otherwise, the result holds trivially. Now, using the triangle inequality and (4.3.12), we can obtain

$$\frac{d(F(\varkappa,y), g\varkappa) + d(F(y,\varkappa), gy)}{2}$$

$$\leq \frac{d\big(F(\varkappa,y), F(\varkappa_n,y_n)\big) + d\big(F(y,\varkappa), F(y_n,\varkappa_n)\big)}{2} + \frac{d(F(\varkappa_n,y_n), g\varkappa) + d(F(y_n,\varkappa_n), gy)}{2}$$

$$= \frac{d\big(F(\varkappa,y), F(\varkappa_n,y_n)\big) + d\big(F(y,\varkappa), F(y_n,\varkappa_n)\big)}{2} + \frac{d(g\varkappa_{n+1}, g\varkappa) + d(gy_{n+1}, gy)}{2}$$

$$\leq \alpha\big((g\varkappa_n, gy_n), (g\varkappa, gy)\big)\left(\frac{d(F(\varkappa,y), F(\varkappa_n,y_n)) + d(F(y,\varkappa), F(y_n,\varkappa_n))}{2}\right) + \frac{d(g\varkappa_{n+1}, g\varkappa) + d(gy_{n+1}, gy)}{2}$$

$$\leq \psi\left(\frac{d(g\varkappa_n, g\varkappa) + d(gy_n, gy)}{2}\right) + \frac{d(g\varkappa_{n+1}, g\varkappa) + d(gy_{n+1}, gy)}{2}$$

$$< \frac{d(g\varkappa_n, g\varkappa) + d(gy_n, gy)}{2} + \frac{d(g\varkappa_{n+1}, g\varkappa) + d(gy_{n+1}, gy)}{2}, \qquad \text{(since } \psi(t) < t \text{ for } t > 0)$$

then, on letting $n \to \infty$, we can obtain $d(F(\varkappa, y), g\varkappa) = 0 = d(F(y, \varkappa), gy)$.

Hence, $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$. This completes our proof.

Now, we give an example in support of Theorem 4.3.2 as follows:

Example 4.3.2. Let us consider the POCMS $(X, \preccurlyeq, d)$ with $X = [0, 1]$, the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$ and $d(x, y) = |\varkappa - y|$ for all $\varkappa, y \in X$. Define the functions $F: X \times X \to X$ and $g: X \to X$ by

$$F(\varkappa, y) = \begin{cases} \frac{\varkappa^2 - y^2}{4}, & \text{if } \varkappa \geq y \\ 0, & \text{if } \varkappa < y \end{cases} \quad \text{and} \quad g\varkappa = \varkappa^2, \text{ for all } \varkappa, y \text{ in } X, \text{ respectively.}$$

Then, F has the MgMP, $(g(X), d)$ is complete and $F(X \times X) \subseteq g(X)$.

Let the function $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ be defined by

$$\alpha\big((\varkappa, y), (u, v)\big) = \begin{cases} 1, & \text{if } \varkappa \geq u,\ y \leq v,\ \text{or } \varkappa \leq u,\ y \geq v, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $(X, \preccurlyeq, d)$ is α-regular space. Let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(t) = \frac{t}{3}$, for $t \in \mathbb{R}^+$. Further, choosing $\varkappa_0 = 0$ and $y_0 = c\ (> 0)$ in X, we have $g\varkappa_0 = 0 = F(0, c) = F(\varkappa_0, y_0)$ and $gy_0 = c^2 \geq \frac{c^2}{4} = F(c, 0) = F(y_0, \varkappa_0)$. Clearly, F is (α) – admissible w.r.t. g. Next, we show that F is (α, ψ) – weak contraction w.r.t. g. If $\alpha\big((\varkappa, y), (u, v)\big) = 0$, the result holds trivially.

Suppose $\alpha\big((\varkappa, y), (u, v)\big) = 1$. We take $\varkappa, y, u, v \in X$, such that $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$, that is, $\varkappa^2 \geq u^2$ and $y^2 \leq v^2$. We consider the following cases:

Case 1: $\varkappa \geq y$, $u \geq v$.

Then, $\alpha\big((\varkappa, y), (u, v)\big)\left(\frac{d(F(\varkappa,y), F(u,v)) + d(F(y,\varkappa), F(v,u))}{2}\right)$

$$= \frac{d(F(\varkappa,y), F(u,v)) + d(F(y,\varkappa), F(v,u))}{2} = \frac{d(F(\varkappa,y), F(u,v)) + d(0,0)}{2} = \frac{1}{2}\, d\left(\frac{\varkappa^2 - y^2}{4}, \frac{u^2 - v^2}{4}\right)$$

$$= \frac{1}{2}\left|\frac{\varkappa^2 - y^2}{4} - \frac{u^2 - v^2}{4}\right| = \frac{1}{2}\left|\frac{(\varkappa^2 - u^2) + (v^2 - y^2)}{4}\right| = \frac{1}{4}\left\{\frac{(\varkappa^2 - u^2) + (v^2 - y^2)}{2}\right\}$$

$$\leq \frac{1}{3}\left\{\frac{(\varkappa^2 - u^2) + (v^2 - y^2)}{2}\right\} = \frac{1}{3}\left\{\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right\} = \psi\left(\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right).$$

Case 2: $\varkappa \geq y$, $u < v$.

Then, $\alpha\big((\varkappa, y), (u,v)\big)\left(\frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}\right) = \frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}$

$= \frac{1}{2}\left\{d\left(\frac{\varkappa^2-y^2}{4}, 0\right) + d\left(0, \frac{v^2-u^2}{4}\right)\right\} = \frac{1}{2}\left\{\left(\frac{\varkappa^2-y^2}{4}\right) + \left(\frac{v^2-u^2}{4}\right)\right\} = \frac{1}{2}\left\{\left(\frac{\varkappa^2-u^2}{4}\right) + \left(\frac{v^2-y^2}{4}\right)\right\}$

$\leq \frac{1}{3}\left\{\frac{(\varkappa^2-u^2) + (v^2-y^2)}{2}\right\} = \frac{1}{3}\left\{\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right\} = \psi\left(\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right).$

Case 3: $\varkappa < y$, $u \geq v$.

Then, $\alpha\big((\varkappa, y), (u,v)\big)\left(\frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}\right)$

$= \frac{d\big(F(\varkappa,y), F(u,v)\big) + d(F(y,\varkappa), F(v,u))}{2} = \frac{1}{2}\left\{d\left(0, \frac{u^2-v^2}{4}\right) + d\left(\frac{y^2-\varkappa^2}{4}, 0\right)\right\}$

$= \frac{1}{2}\left\{\left(\frac{u^2-v^2}{4}\right) + \left(\frac{y^2-\varkappa^2}{4}\right)\right\} = \frac{1}{2}\left\{\frac{-(\varkappa^2-u^2) - (v^2-y^2)}{4}\right\} \leq \frac{1}{4}\left\{\frac{(\varkappa^2-u^2) + (v^2-y^2)}{2}\right\}$

$\leq \frac{1}{3}\left\{\frac{(\varkappa^2-u^2) + (v^2-y^2)}{2}\right\} = \frac{1}{3}\left\{\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right\} = \psi\left(\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right).$

Case 4: $\varkappa < y$, $u < v$.

Then, $\alpha\big((\varkappa, y), (u,v)\big)\left(\frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}\right)$

$= \frac{d\big(F(\varkappa,y), F(u,v)\big) + d(F(y,\varkappa), F(v,u))}{2} = \frac{d(0,0) + d\big(F(y,\varkappa), F(v,u)\big)}{2} = \frac{1}{2}d\left(\frac{y^2-\varkappa^2}{4}, \frac{v^2-u^2}{4}\right)$

$= \frac{1}{2}\left|\frac{y^2-\varkappa^2}{4} - \frac{v^2-u^2}{4}\right| = \frac{1}{2}\left|\frac{-(\varkappa^2-u^2)-(v^2-y^2)}{4}\right| = \frac{1}{2}\left\{\frac{|(\varkappa^2-u^2)+(v^2-y^2)|}{4}\right\} = \frac{1}{2}\left\{\frac{(\varkappa^2-u^2)+(v^2-y^2)}{4}\right\}$

$\leq \frac{1}{3}\left\{\frac{(\varkappa^2-u^2)+(v^2-y^2)}{2}\right\} = \frac{1}{3}\left\{\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right\} = \psi\left(\frac{d(g\varkappa, gu) + d(gv, gy)}{2}\right).$

Thus, F is (α, ψ) – weak contraction w.r.t. g. Therefore, all the conditions of Theorem 4.3.2 are satisfied and (0, 0) is a coupled coincidence point of F and g.

In Theorems 4.3.1 and 4.3.2, considering g to be the identity mapping on X, we have the following result:

Theorem 4.3.3. Let (X, ≼, d) be a POCMS and F: X × X → X be a mapping having MMP on X. Assume that there exist functions α: $X^2 \times X^2 \to \mathbb{R}^+$ and $\psi \in$ CCF-Ψ such that F is (α, ψ) – weak contraction. Also assume that

(viii) F is (α) – admissible;

(ix) there exist $\varkappa_0, y_0 \in X$ such that

$$\alpha\big((\varkappa_0, y_0), (F(\varkappa_0, y_0), F(y_0, \varkappa_0))\big) \geq 1;$$

(x) F is continuous, or (X, ≼, d) is α - regular.

If in the hypothesis (ix), the elements $\varkappa_0, y_0 \in X$ be such that $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(y_0, \varkappa_0)$, then F has a coupled fixed point in X.

Remark 4.3.1. In Theorem 4.3.1 (and in Theorem 4.3.2, respectively), the hypothesis (ii) can be replaced by the following hypothesis:

(xi) there exist $\varkappa_0$, $y_0 \in X$ such that $\alpha\big((gy_0, g\varkappa_0), (F(y_0, \varkappa_0), F(\varkappa_0, y_0))\big) \geq 1$ with $g\varkappa_0 \succcurlyeq F(\varkappa_0, y_0)$ and $gy_0 \preccurlyeq F(y_0, \varkappa_0)$.

Similarly, in Theorem 4.3.3, the hypothesis (ix) can be replaced by the following hypothesis:

(xii) there exist $\varkappa_0$, $y_0 \in X$ such that $\alpha\big((y_0, \varkappa_0), (F(y_0, \varkappa_0), F(\varkappa_0, y_0))\big) \geq 1$ with $\varkappa_0 \succcurlyeq F(\varkappa_0, y_0)$ and $y_0 \preccurlyeq F(y_0, \varkappa_0)$.

Remark 4.3.2. Theorem 4.3.3 along with the Remark 4.3.1 improves Theorem 4.1.3 (Karapinar and Agarwal [158]). Interestingly, Theorem 4.3.3 requires only one of the following conditions:

(a) $\alpha\big((\varkappa_0, y_0), (F(\varkappa_0, y_0), F(y_0, \varkappa_0))\big) \geq 1$, or

(b) $\alpha\big((y_0, \varkappa_0), (F(y_0, \varkappa_0), F(\varkappa_0, y_0))\big) \geq 1$,

to produce the coupled fixed point of the mapping F rather than considering both the conditions (a) and (b), both of these conditions are considered in Theorems 4.1.2 and 4.1.3.

Next, we give an example to show that Theorem 4.3.3 is more general than Theorem 2.1.14 (Bhaskar and Lakshmikantham [55]) and Theorem 4.1.2 (Mursaleen et al. [157]).

Example 4.3.3. Let us consider the POCMS $(X, \preccurlyeq, d)$ with $X = \mathbb{R}$, the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$ and $d(\varkappa, y) = |\varkappa - y|$ for all $\varkappa, y \in X$. Consider the mapping $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ defined as

$$\alpha\big((\varkappa, y), (u, v)\big) = \begin{cases} 1, & \text{if } \varkappa \geq u,\ y \leq v, \text{ or } \varkappa \leq u,\ y \geq v, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(t) = \frac{7}{10} t$, for $t \in \mathbb{R}^+$. Also, define $F: X \times X \to X$ by $F(\varkappa, y) = \frac{6\varkappa - y}{10}$ for $\varkappa, y \in X$. Then, F is continuous, (α) – admissible and has the MMP. We now show that F is (α, ψ) – weak contraction but does not satisfy any of the conditions (2.1.14) of Theorem 2.1.14 and (4.1.7) of Theorem 4.1.2, so that, Theorems 2.1.14 and 4.1.2 do not hold here.

Let, there exists some $k \in [0, 1)$ such that the condition (2.1.14) holds. Then, we have

$$d(F(\varkappa, y), F(u, v)) \leq \frac{k}{2}\,[d(\varkappa, u) + d(y, v)],$$

that is, $$\left|\frac{6\varkappa - y}{10} - \frac{6u - v}{10}\right| \leq \frac{k}{2}\,\{|\varkappa - u| + |y - v|\},\ \varkappa \geq u \text{ and } y \leq v,$$

from which, for $y = v$, we obtain

$$\frac{3}{5}\,|\varkappa - u| \leq \frac{k}{2}\,|\varkappa - u|,\ \varkappa \geq u,$$

which for $\varkappa > u$ implies that $k > 1$, a contradiction, since $k \in [0, 1)$. Therefore, Ғ does not satisfy (2.1.14).

Also, the condition (4.1.7) is not satisfied. On the contrary, suppose there exists some $\psi \in$ CCF-Ψ such that condition (4.1.7) holds. Then, we have

$$\alpha\big((\varkappa, y), (u,v)\big)\, d\big(F(\varkappa, y), F(u,v)\big) \leq \psi\left(\frac{d(\varkappa,u) + d(y,v)}{2}\right)$$

holds for all $\varkappa \geq u$ and $y \leq v$. Taking $\varkappa \neq u$, $y = v$ and $t = |\varkappa - u| > 0$ in the last inequality, we get $\frac{3}{5}t = \frac{3|\varkappa-u|}{5} \leq \psi\left(\frac{|\varkappa-u|}{2}\right) = \psi\left(\frac{t}{2}\right)$, then since $\psi(t) < t$ for $t > 0$, we have $\frac{3}{5}t \leq \psi\left(\frac{t}{2}\right) < \frac{t}{2}$, a contradiction. Therefore, Ғ does not satisfy (4.1.7).

Next, we show that Ғ is (α, ψ) – weak contraction. If $\alpha\big((\varkappa, y), (u,v)\big) = 0$, then the result holds trivially. Suppose that $\alpha\big((\varkappa, y), (u,v)\big) = 1$. W.L.O.G., assume that $\varkappa \geq u$ and $y \leq v$. Then, we get

$$\alpha\big((\varkappa, y), (u,v)\big)\left(\frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}\right) = \frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2}$$

$$= \frac{1}{20}\,|6(\varkappa - u) - (y - v)| + \frac{1}{20}\,|6(y - v) - (\varkappa - u)|$$

$$\leq \frac{7}{10}\left(\frac{|\varkappa-u| + |y-v|}{2}\right) = \psi\left(\frac{d(\varkappa,u) + d(y,v)}{2}\right).$$

Further, choosing $\varkappa_0 = -1$ and $y_0 = 1$ in X such that $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(y_0, \varkappa_0)$. By Theorem 4.3.3 we obtain that Ғ has a coupled fixed point (0, 0) but Theorems 2.1.14 and 4.1.2 cannot be applied to Ғ in this example.

Theorem 4.3.4. Let $(X, \preccurlyeq, d)$ be a POCMS and $F: X \times X \to X$, $g: X \to X$ be two mappings such that Ғ has MgMP on X. Assume that there exist some $\psi \in$ CCF-Ψ such that

$$\frac{d\big(F(\varkappa,y), F(u,v)\big) + d\big(F(y,\varkappa), F(v,u)\big)}{2} \leq \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right), \tag{4.3.13}$$

for all $\varkappa, y, u, v \in X$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$ (or $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$). Also, assume the following conditions:

(xiii) (Ғ, g) is compatible;

(xiv) Ғ and g both are continuous;

(xv) $F(X \times X) \subseteq g(X)$;

(xvi) there exist $\varkappa_0, y_0 \in X$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$.

Then, Ғ and g have a coupled coincidence point in X.

Proof. Consider the mapping $\alpha: X^2 \times X^2 \to \mathbb{R}^+$ defined by

$$\alpha\big((\varkappa, y), (u,v)\big) = \begin{cases} 1, & \text{if } \varkappa \succcurlyeq u,\ y \preccurlyeq v,\ \text{or } \varkappa \preccurlyeq u,\ y \succcurlyeq v, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.14}$$

then, using assumption (xvi), we get $\alpha\big((g\varkappa_0, gy_0), (Ғ(\varkappa_0, y_0), Ғ(y_0, \varkappa_0))\big) \geq 1$.

Now, for all $(\varkappa, y), (u, v) \in Ҳ \times Ҳ$,

$\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1 \Longrightarrow g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$ or $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$.

Then, since Ғ has MgMP, we obtain

$Ғ(\varkappa, y) \succcurlyeq Ғ(u, v)$ and $Ғ(y, \varkappa) \preccurlyeq Ғ(v, u)$ or $Ғ(y, \varkappa) \succcurlyeq Ғ(v, u)$ and $Ғ(\varkappa, y) \preccurlyeq Ғ(u, v)$

which implies $\alpha\Big(\big(Ғ(\varkappa, y), Ғ(y,\varkappa)\big), \big(Ғ(u,v), Ғ(v, u)\big)\Big) \geq 1$.

Hence, Ғ is (α) – admissible w.r.t. g. Further, by (4.3.13) and (4.3.14), Ғ is (α, ψ) – weak contraction w.r.t. g. Then, by Theorem 4.3.1 we can obtain the existence of coupled coincidence point of Ғ and g.

Remark 4.3.3. Result similar to Theorem 4.3.4 can be deduced from Theorem 4.3.2 for α - regular spaces.

Remark 4.3.4. (i) On taking $\psi(f) = kf$, with $k \in [0, 1)$ in Theorem 4.3.4, we can obtain contraction (3.2.23) of Corollary 3.2.1. Hence, Theorem 4.3.4 along with Remark 4.3.3 provides a generalization of Corollary 3.2.1 (Jain et al. [159]).

(ii) On taking $\psi(f) = kf$, where $k \in [0, 1)$ and g to be the identity mapping on Ҳ in Theorem 4.3.4, the contraction (4.3.13) becomes (3.1.1). Hence, Theorem 4.3.4 along with Remark 4.3.3 provides a generalization of Theorem 3.1.1 (Berinde [149]).

Coupled Common Fixed Points

Now, we establish the existence and uniqueness of the coupled common fixed point under the hypotheses of Theorem 4.3.1 with some additional assumption.

Theorem 4.3.5. In addition to the hypotheses of Theorem 4.3.1, suppose that for every $(\varkappa, y), (\varkappa^*, y^*) \in Ҳ \times Ҳ$, there exists some $(u, v) \in Ҳ \times Ҳ$ such that

$$\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1 \quad \text{and} \quad \alpha\big((g\varkappa^*, gy^*), (gu, gv)\big) \geq 1.$$

Also, assume that (gu, gv) is comparable to $(g\varkappa, gy)$ and $(g\varkappa^*, gy^*)$. Then, Ғ and g have a unique coupled common fixed point in Ҳ.

Proof. By Theorem 4.3.1, the set of coupled coincidences is non-empty. In order to prove the result, we first show that if $(\varkappa, y)$ and $(\varkappa^*, y^*)$ are coupled coincidence points, then

$$g\varkappa = g\varkappa^* \text{ and } gy = gy^*. \tag{4.3.15}$$

Now, by assumption, there exists some $(u, v) \in Ҳ \times Ҳ$ such that

$$\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1, \quad \alpha\big((g\varkappa^*, gy^*), (gu, gv)\big) \geq 1, \tag{4.3.16}$$

and (gu, gv) is comparable to ($g\varkappa$, gy) and ($g\varkappa^*$, gy^*). Take $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$, $gv_1 = F(v_0, u_0)$.

Then, as in the proof of Theorem 4.3.1, using induction, we can define the sequences $\{gu_n\}$ and $\{gv_n\}$ with $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$.

Take $\varkappa_0 = \varkappa$, $y_0 = y$, $\varkappa_0^* = \varkappa^*$, $y_0^* = y^*$ and on the same way define the sequences $\{g\varkappa_n\}$, $\{gy_n\}$ and $\{g\varkappa_n^*\}$, $\{gy_n^*\}$. Then, we can obtain that

$$g\varkappa_{n+1} = F(\varkappa_n, y_n),\ gy_{n+1} = F(y_n, \varkappa_n)$$

and $$g\varkappa_{n+1}^* = F(\varkappa_n^*, y_n^*),\ gy_{n+1}^* = F(y_n^*, \varkappa_n^*) \text{ for all } n \geq 0.$$

Since (gu, gv) is comparable with ($g\varkappa$, gy), we assume that $(g\varkappa, gy) \succcurlyeq (gu, gv) = (gu_0, gv_0)$. By proof of Theorem 4.3.1, we can inductively obtain that $(g\varkappa, gy) \succcurlyeq (gu_n, gv_n)$ for all $n \geq 0$. Since F is (α) - admissible w.r.t. g, so that using (4.3.16), we have $\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1 \Longrightarrow \alpha\Big(\big(F(\varkappa, y), F(y,\varkappa)\big), \big(F(u,v), F(v, u)\big)\Big) \geq 1.$

Since $u = u_0$ and $v = v_0$, we get

$$\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1 \Longrightarrow \alpha\Big(\big(F(\varkappa, y), F(y,\varkappa)\big), \big(F(u_0, v_0), F(v_0, u_0)\big)\Big) \geq 1.$$

Therefore, $\alpha\big((g\varkappa, gy), (gu, gv)\big) \geq 1 \Longrightarrow \alpha\big((g\varkappa, gy), (gu_1, gv_1)\big) \geq 1$.

Then, by mathematical induction, we get

$$\alpha\big((g\varkappa, gy), (gu_n, gv_n)\big) \geq 1, \tag{4.3.17}$$

for all $n \in \mathbb{N}$. From (4.3.16) and (4.3.17), we get

$$\begin{aligned}\frac{d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1})}{2} &= \frac{d\big(F(\varkappa,y), F(u_n,v_n)\big) + d\big(F(y,\varkappa), F(v_n,u_n)\big)}{2}\\ &\leq \alpha\big((g\varkappa, gy), (gu_n, gv_n)\big)\frac{d\big(F(\varkappa,y), F(u_n,v_n)\big) + d\big(F(y,\varkappa), F(v_n,u_n)\big)}{2}\\ &\leq \psi\left(\frac{d(g\varkappa, gu_n) + d(gy, gv_n)}{2}\right).\end{aligned}$$

Therefore,

$$\frac{d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1})}{2} \leq \psi^n\left(\frac{d(g\varkappa, gu_0) + d(gy, gv_0)}{2}\right), \tag{4.3.18}$$

for each $n \geq 1$. Taking $n \to \infty$ in (4.3.18), we obtain

$$\lim_{n\to\infty}[d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1})] = 0.$$

which implies

$$\lim_{n\to\infty} d(g\varkappa, gu_{n+1}) = \lim_{n\to\infty} d(gy, gv_{n+1}) = 0. \tag{4.3.19}$$

Similarly, we get

$$\lim_{n\to\infty} d(g\varkappa^*, gu_{n+1}) = \lim_{n\to\infty} d(gy^*, gv_{n+1}) = 0. \tag{4.3.20}$$

Using (4.3.19) and (4.3.20), we can obtain $g\varkappa = g\varkappa^*$ and $gy = gy^*$. Hence, we have proved (4.3.15). Now, since $g\varkappa = F(\varkappa, y)$, $gy = F(y, \varkappa)$ and the pair (F, g) is compatible, then by Lemma 3.2.1, it follows that

$$gg\varkappa = gF(\varkappa, y) = F(g\varkappa, gy) \text{ and } ggy = gF(y, \varkappa) = F(gy, g\varkappa). \quad (4.3.21)$$

Denote $g\varkappa = z$, $gy = w$. Then using (4.3.21), we get

$$gz = F(z, w) \text{ and } gw = F(w, z). \quad (4.3.22)$$

Thus (z, w) is a coupled coincidence point.

Then by (4.3.15) with $\varkappa^* = z$ and $y^* = w$, it follows that $gz = g\varkappa$ and $gw = gy$, so that

$$gz = z,\ gw = w. \quad (4.3.23)$$

Now, using (4.3.22) and (4.3.23), we obtain that

$$z = gz = F(z, w) \text{ and } w = gw = F(w, z).$$

Therefore, (z, w) is a coupled common fixed point of F and g.

For uniqueness, let (s, l) be any coupled common fixed point of F and g, then, using (4.3.15), we have $s = gs = gz = z$ and $l = gl = gw = w$.

Thus, F and g have a unique coupled common fixed point.

Theorem 4.3.6. In addition to the hypotheses of Theorem 4.3.3, assume that for every $(\varkappa, y), (\varkappa^*, y^*) \in X \times X$ there exists some $(u, v) \in X \times X$ comparable to $(\varkappa, y)$ and $(\varkappa^*, y^*)$ such that

$$\alpha\big((\varkappa, y), (u,v)\big) \geq 1 \qquad \text{and} \qquad \alpha\big((\varkappa^*, y^*), (u,v)\big) \geq 1.$$

Then, F has a unique coupled fixed point in X.

Proof. The proof follows easily by taking g to be the identity mapping on X in Theorem 4.3.5.

4.4 APPLICATION TO INTEGRAL EQUATIONS

In this section, as application of the results proved in the earlier sections of this chapter, we give the solution of the integral equations.

Firstly, as application of the results proved in section 4.2, we discuss the existence of solutions for the following system of integral equations:

$$\begin{aligned} \varkappa(t) &= \int_c^d \big(K_1(t,s) + K_2(t,s)\big)\Big(f\big(s,\varkappa(s)\big) + g\big(s, y(s)\big)\Big)\,ds \ +\ \hbar(t), \\ y(t) &= \int_c^d \big(K_1(t,s) + K_2(t,s)\big)\Big(f\big(s, y(s)\big) + g\big(s,\varkappa(s)\big)\Big)\,ds \ +\ \hbar(t), \end{aligned} \quad (4.4.1)$$

$t \in I = [c, d]$.

Let Θ_1 denote the class of functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) φ is increasing;

(ii) for each $\varkappa \geq 0$, there exists some $k \in (0, 1)$ such that $\varphi(\varkappa) \leq (k/2)\varkappa$.

We consider the following assumptions:

Assumption 4.4.1. (i) $K_1(t, s) \geq 0$ and $K_2(t, s) \leq 0$ for all $t, s \in I$;

(ii) there exist $\lambda, \mu > 0$ and $\varphi \in \Theta_1$ such that for all $\varkappa, y \in \mathbb{R}$, $\varkappa > y$,

$$0 < f(t, \varkappa) - f(t, y) \leq \lambda\varphi(\varkappa - y), \quad (4.4.2)$$

and

$$-\mu\varphi(\varkappa - y) \leq g(t, \varkappa) - g(t, y) < 0; \quad (4.4.3)$$

(iii) $(\lambda + \mu) \cdot \sup_{t\in I} \int_c^d \big(K_1(t, s) - K_2(t, s)\big)\, ds \leq 1.$ (4.4.4)

Definition 4.4.1. An element $(\hat{\varkappa}, \hat{y}) \in X \times X$ with $X = C(I, \mathbb{R})$ is called a **coupled lower and upper solution** of the integral equation (4.4.1) if for all $t \in I$, we have

$$\hat{\varkappa}(t) < \int_c^d K_1(t, s) \Big(f\big(s, \hat{\varkappa}(s)\big) + g\big(s, \hat{y}(s)\big)\Big)\, ds$$
$$+ \int_c^d K_2(t, s) \Big(f\big(s, \hat{y}(s)\big) + g\big(s, \hat{\varkappa}(s)\big)\Big)\, ds + h(t)$$

and

$$\hat{y}(t) \geq \int_c^d K_1(t, s) \Big(f\big(s, \hat{y}(s)\big) + g\big(s, \hat{\varkappa}(s)\big)\Big)\, ds$$
$$+ \int_c^d K_2(t, s) \Big(f\big(s, \hat{\varkappa}(s)\big) + g\big(s, \hat{y}(s)\big)\Big)\, ds + h(t).$$

Theorem 4.4.1. Consider the integral equation (4.4.1) with $K_1, K_2 \in C(I \times I, \mathbb{R})$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and $h \in C(I, \mathbb{R})$. Let the Assumption 4.4.1 is satisfied and $(\hat{\varkappa}, \hat{y})$ be the coupled lower and upper solution of (4.4.1). Then, the integral equation (4.4.1) has a solution.

Proof. Consider the following order relation on $X = C(I, \mathbb{R})$:

for $\varkappa, y \in X$, $\quad \varkappa \preccurlyeq y \quad$ iff $\quad \varkappa(t) \leq y(t)$, for $t \in I$.

Also, X is a complete metric space w.r.t. sup metric:

$$d(\varkappa, y) = \sup_{t\in I} |\varkappa(t) - y(t)|, \ \varkappa, y \in C(I, \mathbb{R}).$$

Further, the conditions (i) and (ii) in Corollary 4.2.1 hold.

Also, $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a poset under the following order relation:

$(\varkappa, y), (u, v) \in X \times X$, $(\varkappa, y) \preccurlyeq (u, v)$ iff $\varkappa(t) \leq u(t)$ and $y(t) \geq v(t)$, for $t \in I$.

Define the mapping $F: X \times X \to X$ by

$$F(\varkappa, y)(t) = \int_c^d K_1(t, s) \Big(f\big(s, \varkappa(s)\big) + g\big(s, y(s)\big)\Big)\, ds$$
$$+ \int_c^d K_2(t, s) \Big(f\big(s, y(s)\big) + g\big(s, \varkappa(s)\big)\Big)\, ds + h(t), \quad \text{for } t \in I.$$

Now, we shall show that F has the MSMP.

For, $\varkappa_1 \prec \varkappa_2$, that is, if $\varkappa_1(t) < \varkappa_2(t)$ for $t \in I$, we have

$$F(\varkappa_1, y)(t) - F(\varkappa_2, y)(t) = \int_c^d K_1(t, s) \Big(f\big(s, \varkappa_1(s)\big) + g\big(s, y(s)\big)\Big)\, ds$$

$$+ \int_c^d K_2(t,s)\left(f\big(s,y(s)\big) + g\big(s,\varkappa_1(s)\big)\right) ds + h(t)$$
$$- \int_c^d K_1(t,s)\left(f\big(s,\varkappa_2(s)\big) + g\big(s,y(s)\big)\right) ds$$
$$- \int_c^d K_2(t,s)\left(f\big(s,y(s)\big) + g\big(s,\varkappa_2(s)\big)\right) ds - h(t)$$
$$= \int_c^d K_1(t,s)\left(f\big(s,\varkappa_1(s)\big) - f\big(s,\varkappa_2(s)\big)\right) ds$$
$$+ \int_c^d K_2(t,s)\left(g\big(s,\varkappa_1(s)\big) - g\big(s,\varkappa_2(s)\big)\right) ds < 0,$$

by Assumption 4.4.1.

Therefore, $F(\varkappa_1, y)(t) < F(\varkappa_2, y)(t)$, for all $t \in I$, so that $F(\varkappa_1, y) \prec F(\varkappa_2, y)$.

Similarly, if $y_1 \succ y_2$, we can obtain that $F(\varkappa, y_1) \prec F(\varkappa, y_2)$. Therefore, F has MSMP.

Next, we show that F satisfies (4.2.2).

For, $\varkappa \succcurlyeq u$, $y \preccurlyeq v$ (that is, $\varkappa(t) \geq u(t)$, $y(t) \leq v(t)$ for all $t \in I$), then, we have

$F(\varkappa, y)(t) - F(u, v)(t)$

$$= \int_c^d K_1(t,s)\left(f\big(s,\varkappa(s)\big) + g\big(s,y(s)\big)\right) ds$$
$$+ \int_c^d K_2(t,s)\left(f\big(s,y(s)\big) + g\big(s,\varkappa(s)\big)\right) ds - \int_c^d K_1(t,s)\left(f\big(s,u(s)\big) + g\big(s,v(s)\big)\right) ds$$
$$- \int_c^d K_2(t,s)\left(f\big(s,v(s)\big) + g\big(s,u(s)\big)\right) ds.$$
$$= \int_c^d K_1(t,s)\left(f\big(s,\varkappa(s)\big) - f\big(s,u(s)\big) + g\big(s,y(s)\big) - g\big(s,v(s)\big)\right) ds$$
$$+ \int_c^d K_2(t,s)\left(f\big(s,y(s)\big) - f\big(s,v(s)\big) + g\big(s,\varkappa(s)\big) - g\big(s,u(s)\big)\right) ds$$
$$= \int_c^d K_1(t,s)\left[\left(f\big(s,\varkappa(s)\big) - f\big(s,u(s)\big)\right) - \left(g\big(s,v(s)\big) - g\big(s,y(s)\big)\right)\right] ds$$
$$- \int_c^d K_2(t,s)\left[\left(f\big(s,v(s)\big) - f\big(s,y(s)\big)\right) - \left(g\big(s,\varkappa(s)\big) - g\big(s,u(s)\big)\right)\right] ds$$
$$\leq \int_c^d K_1(t,s)\,[\lambda\varphi(\varkappa(s) - u(s)) + \mu\varphi(v(s) - y(s))] ds$$
$$- \int_c^d K_2(t,s)\,[\lambda\varphi(v(s) - y(s)) + \mu\varphi(\varkappa(s) - u(s))] ds. \qquad (4.4.5)$$

Since φ is an increasing function and $\varkappa \succcurlyeq u$, $y \preccurlyeq v$, we get

$$\varphi(\varkappa(s) - u(s)) \leq \varphi(\sup_{t \in I}|\varkappa(t) - u(t)|) = \varphi\big(d(\varkappa, u)\big)$$

and $\quad \varphi(v(s) - y(s)) \leq \varphi(\sup_{t \in I}|v(t) - y(t)|) = \varphi\big(d(v, y)\big).$

Then, using (4.4.5) and the fact that $K_2(t, s) \leq 0$, we can obtain

$$|F(\varkappa, y)(t) - F(u,v)(t)| \leq \int_c^d K_1(t,s)\left[\lambda\varphi\big(d(\varkappa, u)\big) + \mu\varphi\big(d(v, y)\big)\right] ds$$
$$- \int_c^d K_2(t,s)\left[\lambda\varphi\big(d(v, y)\big) + \mu\varphi\big(d(\varkappa, u)\big)\right] ds. \qquad (4.4.6)$$

Since all the quantities on the right hand side of (4.4.5) are non-negative, so (4.4.6) is satisfied. Similarly, we can get

$$|\text{F}(y,\varkappa)(t) - \text{F}(v,u)(t)| \le \int_c^d K_1(t,s)\left[\lambda\varphi\big(d(v,y)\big) + \mu\varphi\big(d(\varkappa,u)\big)\right]ds$$

$$- \int_c^d K_2(t,s)\left[\lambda\varphi\big(d(\varkappa,u)\big) + \mu\varphi\big(d(v,y)\big)\right]ds. \qquad (4.4.7)$$

Adding (4.4.6) and (4.4.7), dividing by 2 and taking supremum w.r.t. t, then using (4.4.4) we can obtain

$$\frac{d\big(\text{F}(\varkappa,y)+\text{F}(u,v)\big) + d\big(\text{F}(y,\varkappa)+\text{F}(v,u)\big)}{2}$$

$$\le (\lambda+\mu)\sup_{t\in I}\int_c^d \big(K_1(t,s) - K_2(t,s)\big)\,ds \cdot \frac{\varphi\big(d(v,y)\big) + \varphi\big(d(\varkappa,u)\big)}{2}$$

$$\le \frac{\varphi\big(d(v,y)\big) + \varphi\big(d(\varkappa,u)\big)}{2} \le \varphi\big(d(\varkappa,u) + d(v,y)\big) \le (k/2)[d(\varkappa,u) + d(v,y)],$$

by using the definition of φ. Therefore, we can get

$$d\big(\text{F}(\varkappa,y),\text{F}(u,v)\big) + d\big(\text{F}(y,\varkappa),\text{F}(v,u)\big) \le k[d(\varkappa,u) + d(v,y)],$$

which is the contractive condition (4.2.2). Then, by Proposition 4.2.1, F is a generalized symmetric Meir-Keeler type contraction. Finally, suppose ($\hat{\varkappa}$, $\hat{y}$) be a coupled lower and upper solution of the integral equation (4.4.1), then we can obtain $\hat{\varkappa} \prec \text{F}(\hat{\varkappa}, \hat{y})$ and $\hat{y} \succcurlyeq \text{F}(\hat{y}, \hat{\varkappa})$. Then applying Corollary 4.2.1, we get that F has a coupled fixed point ($\varkappa$, y) and therefore, the system (4.4.1) of integral equations has a solution.

Next, as an application of the results proved in section 4.3, we discuss the existence of solutions for the following system of integral equations:

$$\begin{aligned} \varkappa(t) &= \int_c^d \big(K_1(t,s) + K_2(t,s)\big)\Big(f\big(s,\varkappa(s)\big) + g\big(s,y(s)\big)\Big)\,ds + \hbar(t), \\ y(t) &= \int_c^d \big(K_1(t,s) + K_2(t,s)\big)\Big(f\big(s,y(s)\big) + g\big(s,\varkappa(s)\big)\Big)\,ds + \hbar(t), \end{aligned} \qquad (4.4.8)$$

t ∈ I = [c, d].

Let Θ_2 denote the class of functions φ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) φ is increasing;

(ii) for each $\varkappa \ge 0$, there exists some ψ ∈ CCF-Ψ such that $\varphi(\varkappa) \le \psi(\varkappa/2)$.

We consider the following assumptions:

Assumption 4.4.2. (i) $K_1(t, s) \ge 0$ and $K_2(t, s) \le 0$ for all t, s ∈ I;

(ii) there exist λ, μ > 0 and φ ∈ Θ_2 such that for $\varkappa$, y ∈ ℝ, $\varkappa \ge y$,

$$0 \le f(t,\varkappa) - f(t,y) \le \lambda\varphi(\varkappa - y), \qquad (4.4.9)$$

and

$$-\mu\varphi(\varkappa - y) \le g(t,\varkappa) - g(t,y) \le 0; \qquad (4.4.10)$$

(iii) $\qquad (\lambda + \mu) \cdot \sup_{t\in I} \int_c^d \big(K_1(t,s) - K_2(t,s)\big)\, ds \leq 1. \qquad$ (4.4.11)

Definition 4.4.2. An element $(\hat{x}, \hat{y}) \in X \times X$ with $X = C(I, \mathbb{R})$ is called a **coupled lower and upper solution** of the integral equation (4.4.8), if for all $t \in I$, we have

$$\hat{x}(t) \leq \int_c^d K_1(t,s)\Big(f\big(s,\hat{x}(s)\big) + g\big(s,\hat{y}(s)\big)\Big)\, ds + \int_c^d K_2(t,s)\Big(f\big(s,\hat{y}(s)\big) + g\big(s,\hat{x}(s)\big)\Big)\, ds + h(t),$$

and
$$\hat{y}(t) \geq \int_c^d K_1(t,s)\Big(f\big(s,\hat{y}(s)\big) + g\big(s,\hat{x}(s)\big)\Big)\, ds + \int_c^d K_2(t,s)\Big(f\big(s,\hat{x}(s)\big) + g\big(s,\hat{y}(s)\big)\Big)\, ds + h(t).$$

Theorem 4.4.2. Consider the integral equation (4.4.8) with $K_1, K_2 \in C(I \times I, \mathbb{R})$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and $h \in C(I, \mathbb{R})$. Let the Assumption 4.4.2 is satisfied and $(\hat{x}, \hat{y})$ be the coupled lower and upper solution of (4.4.8). Then, the integral equation (4.4.8) has a solution.

Proof. Consider the following order relation on $X = C(I, \mathbb{R})$:

For $x, y \in X$, $\qquad x \preccurlyeq y \;$ iff $\; x(t) \leq y(t)$, for $t \in I$.

Also, X is a complete metric space w.r.t. sup metric:

$$d(x, y) = \sup_{t\in I} |x(t) - y(t)|, \text{ for } x, y \in C(I, \mathbb{R}).$$

Also, $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a poset under the following order relation:

$(x, y), (u, v) \in X \times X$, $(x, y) \preccurlyeq (u, v)$ iff $x(t) \leq u(t)$ and $y(t) \geq v(t)$, for $t \in I$.

Define $F: X \times X \to X$ by

$$F(x, y)(t) = \int_c^d K_1(t,s)\Big(f\big(s,x(s)\big) + g\big(s,y(s)\big)\Big)\, ds + \int_c^d K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x(s)\big)\Big)\, ds + h(t), \text{ for all } t \in I.$$

Now, we show that F has MMP.

For, $x_1 \prec x_2$, that is, $x_1(t) \leq x_2(t)$ for all $t \in I$, we have

$$\begin{aligned} F(x_1, y)(t) - F(x_2, y)(t) &= \int_c^d K_1(t,s)\Big(f\big(s,x_1(s)\big) + g\big(s,y(s)\big)\Big)\, ds \\ &\quad + \int_c^d K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x_1(s)\big)\Big)\, ds + h(t) \\ &\quad - \int_c^d K_1(t,s)\Big(f\big(s,x_2(s)\big) + g\big(s,y(s)\big)\Big)\, ds \\ &\quad - \int_c^d K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x_2(s)\big)\Big)\, ds - h(t) \\ &= \int_c^d K_1(t,s)\Big(f\big(s,x_1(s)\big) - f\big(s,x_2(s)\big)\Big)\, ds \\ &\quad + \int_c^d K_2(t,s)\Big(g\big(s,x_1(s)\big) - g\big(s,x_2(s)\big)\Big)\, ds \leq 0, \end{aligned}$$

by Assumption 4.4.2.

Therefore, $\mathrm{F}(\varkappa_1\ y)(\mathfrak{t}) \leq \mathrm{F}(\varkappa_2, y)(\mathfrak{t})$ for $\mathfrak{t} \in I$, so that $\mathrm{F}(\varkappa_1, y) \preccurlyeq \mathrm{F}(\varkappa_2, y)$.

Similarly, for $y_1 \geq y_2$, we can obtain that $\mathrm{F}(\varkappa, y_1) \preccurlyeq \mathrm{F}(\varkappa, y_2)$. Therefore, F satisfies MMP.

Next, we show that F is (α, ψ) – weak contraction for some $\alpha: \mathfrak{X}^2 \times \mathfrak{X}^2 \to \mathbb{R}^+$.

For, $\varkappa \succcurlyeq u$, $y \preccurlyeq v$ (that is, $\varkappa(\mathfrak{t}) \geq u(\mathfrak{t})$, $y(\mathfrak{t}) \leq v(\mathfrak{t})$ for all $\mathfrak{t} \in I$), then, we have

$$\mathrm{F}(\varkappa, y)(\mathfrak{t}) - \mathrm{F}(u, v)(\mathfrak{t}) = \int_c^d K_1(\mathfrak{t}, s)\left(f(s, \varkappa(s)) + g(s, y(s))\right) ds$$

$$+ \int_c^d K_2(\mathfrak{t}, s)\left(f(s, y(s)) + g(s, \varkappa(s))\right) ds$$

$$- \int_c^d K_1(\mathfrak{t}, s)\left(f(s, u(s)) + g(s, v(s))\right) ds$$

$$- \int_c^d K_2(\mathfrak{t}, s)\left(f(s, v(s)) + g(s, u(s))\right) ds.$$

$$= \int_c^d K_1(\mathfrak{t}, s)\left(f(s, \varkappa(s)) - f(s, u(s)) + g(s, y(s)) - g(s, v(s))\right) ds$$

$$+ \int_c^d K_2(\mathfrak{t}, s)\left(f(s, y(s)) - f(s, v(s)) + g(s, \varkappa(s)) - g(s, u(s))\right) ds$$

$$= \int_c^d K_1(\mathfrak{t}, s)\left[\left(f(s, \varkappa(s)) - f(s, u(s))\right) - \left(g(s, v(s)) - g(s, y(s))\right)\right] ds$$

$$- \int_c^d K_2(\mathfrak{t}, s)\left[\left(f(s, v(s)) - f(s, y(s))\right) - \left(g(s, \varkappa(s)) - g(s, u(s))\right)\right] ds$$

$$\leq \int_c^d K_1(\mathfrak{t}, s)\left[\lambda\varphi(\varkappa(s) - u(s)) + \mu\varphi(v(s) - y(s))\right] ds$$

$$- \int_c^d K_2(\mathfrak{t}, s)\left[\lambda\varphi(v(s) - y(s)) + \mu\varphi(\varkappa(s) - u(s))\right] ds. \qquad (4.4.12)$$

Since, φ is an increasing function and $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we get

$$\varphi(\varkappa(s) - u(s)) \leq \varphi(\sup_{\mathfrak{t} \in I}|\varkappa(\mathfrak{t}) - u(\mathfrak{t})|) = \varphi(\mathfrak{d}(\varkappa, u)),$$

and $$\varphi(v(s) - y(s)) \leq \varphi(\sup_{\mathfrak{t} \in I}|v(\mathfrak{t}) - y(\mathfrak{t})|) = \varphi(\mathfrak{d}(v, y)).$$

Then, using (4.4.12) and the fact that $K_2(\mathfrak{t}, s) \leq 0$, we can obtain

$$|\mathrm{F}(\varkappa, y)(\mathfrak{t}) - \mathrm{F}(u, v)(\mathfrak{t})| \leq \int_c^d K_1(\mathfrak{t}, s)\left[\lambda\varphi(\mathfrak{d}(\varkappa, u)) + \mu\varphi(\mathfrak{d}(v, y))\right] ds$$

$$- \int_c^d K_2(\mathfrak{t}, s)\left[\lambda\varphi(\mathfrak{d}(v, y)) + \mu\varphi(\mathfrak{d}(\varkappa, u))\right] ds. \qquad (4.4.13)$$

Since all the quantities on the right hand side of (4.4.12) are non-negative, hence (4.4.13) holds. Similarly, we can obtain

$$|\mathrm{F}(y, \varkappa)(\mathfrak{t}) - \mathrm{F}(v, u)(\mathfrak{t})| \leq \int_c^d K_1(\mathfrak{t}, s)\left[\lambda\varphi(\mathfrak{d}(v, y)) + \mu\varphi(\mathfrak{d}(\varkappa, u))\right] ds$$

$$- \int_c^d K_2(\mathfrak{t}, s)\left[\lambda\varphi(\mathfrak{d}(\varkappa, u)) + \mu\varphi(\mathfrak{d}(v, y))\right] ds. \qquad (4.4.14)$$

Adding (4.4.13) and (4.4.14), dividing by 2, and taking the supremum w.r.t. $\mathfrak{t}$, then using (4.4.11), we obtain that

$$\frac{ḑ(Ғ(\varkappa,y),Ғ(u,v)) + ḑ(Ғ(y,\varkappa),Ғ(v,u))}{2}$$

$$\leq (\lambda+\mu)\sup_{t\in I}\int_a^b\big(K_1(t,s)-K_2(t,s)\big)\,ds\cdot\frac{\varphi(ḑ(v,y))+\varphi(ḑ(\varkappa,u))}{2}$$

$$\leq \frac{\varphi(ḑ(v,y))+\varphi(ḑ(\varkappa,u))}{2}.$$

Since φ is an increasing function, we have

$$\varphi\big(ḑ(\varkappa,u)\big)\leq\varphi\big(ḑ(\varkappa,u)+ḑ(v,y)\big),\qquad \varphi\big(ḑ(v,y)\big)\leq\varphi\big(ḑ(\varkappa,u)+ḑ(v,y)\big)$$

and hence, using the definition of φ, we get

$$\frac{\varphi(ḑ(v,y))+\varphi(ḑ(\varkappa,u))}{2}\leq\varphi\big(ḑ(\varkappa,u)+ḑ(v,y)\big)\leq\psi\left(\frac{ḑ(\varkappa,u)+ḑ(v,y)}{2}\right).$$

Therefore, we can obtain

$$\frac{ḑ(Ғ(\varkappa,y),Ғ(u,v)) + ḑ(Ғ(y,\varkappa),Ғ(v,u))}{2}\leq\psi\left(\frac{ḑ(\varkappa,u)+ḑ(v,y)}{2}\right). \tag{4.4.15}$$

Define the mapping α: $X^2\times X^2\to\mathbb{R}^+$ by $\alpha\big((\varkappa,y),(u,v)\big)=1$, if the pairs $(\varkappa, y)$ and (u, v) are comparable w.r.t. the ordering in $X\times X$ and $\alpha\big((\varkappa,y),(u,v)\big)=0$, otherwise. Now, using the definition of α and the MMP of Ғ, we can get

$$\alpha\Big(\big(Ғ(\varkappa,y),Ғ(y,\varkappa)\big),\big(Ғ(u,v),Ғ(v,u)\big)\Big)\geq 1.$$

Therefore, Ғ is (α) – admissible. Further, using the definition of α and (4.4.15), Ғ is (α,ψ) – weak contraction.

Also, suppose that $\{u_n\}$ and $\{v_n\}$ be the two convergent sequences in X converging to u and v, respectively. Let $u_n\preccurlyeq u_{n+1}$ and $v_n\succcurlyeq v_{n+1}$ for all n > 0.

Then, by the definition of α, we get $\alpha\big((u_n,v_n),(u_{n+1},v_{n+1})\big)\geq 1$, for all n > 0. Now, $\{u_n\}$ is an increasing sequence in X converging to u, so we have $u_n\preccurlyeq u$ for all n. Also, $\{v_n\}$ is a decreasing sequence in X converging to v, so we have $v\preccurlyeq v_n$ for all n. Again, using the definition of α, we get $\alpha\big((u_n,v_n),(u,v)\big)\geq 1$ for all n. Therefore, the space $(X,\preccurlyeq,ḑ)$ is α - regular.

Also, for any $\varkappa$, y ∈ X and each t ∈ I, the max$\{\varkappa(t), y(t)\}$ and min$\{\varkappa(t), y(t)\}$ are in X and are the upper and lower bounds of $\varkappa$ and y, respectively. Therefore, for every $(\varkappa, y), (u, v)\in X\times X$, there exists a $(\max\{\varkappa, u\}, \min\{y, v\})\in X\times X$ which is comparable to $(\varkappa, y)$ and (u, v). Then, by definition of α, we obtain

$$\alpha\big((\varkappa,y),(\max\{\varkappa,u\},\min\{y,v\})\big)\geq 1\ \text{ and }\ \alpha\big((u,v),(\max\{\varkappa,u\},\min\{y,v\})\big)\geq 1.$$

Finally, let $(\hat{\varkappa},\hat{y})$ be a coupled lower and upper solution of the integral equation (4.4.8), then, we can obtain that $\hat{\varkappa}\preccurlyeq Ғ(\hat{\varkappa},\hat{y})$ and $\hat{y}\succcurlyeq Ғ(\hat{y},\hat{\varkappa})$. Then, using the definition of α, we have $\alpha\big((\hat{\varkappa},\hat{y}),(Ғ(\hat{\varkappa},\hat{y}),Ғ(\hat{y},\hat{\varkappa}))\big)\geq 1$. Now, by Theorems 4.3.3 and 4.3.6 we can obtain that Ғ has a unique coupled fixed point (ϰ, y) and therefore, the system (4.4.8) of integral equations has a unique solution.

4.5 APPLICATION TO RESULTS OF INTEGRAL TYPE

In this section, we discuss an application of the results proved in section 4.2 in terms of the integrals.

Theorem 4.5.1. Let (Ⴟ, ≼, ȡ) be a POMS and Ғ: Ⴟ × Ⴟ → Ⴟ, g: Ⴟ → Ⴟ be the two given mappings. Suppose there exists a function θ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(I) θ(0) = 0 and θ(f) > 0 for any f > 0;

(II) θ is right continuous and increasing;

(III) for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all ϰ, y, u, v ∈ Ⴟ with gϰ ≼ gu and gy ≽ gv,

$$\varepsilon \leq \theta\left(\frac{1}{2}[ȡ(gϰ, gu) + ȡ(gy, gv)]\right) < \varepsilon + \delta(\varepsilon),$$

implies $$\theta\left(\frac{1}{2}\left[ȡ\big(Ғ(ϰ, y), Ғ(u, v)\big) + ȡ\big(Ғ(y, ϰ), Ғ(v, u)\big)\right]\right) < \varepsilon. \qquad (4.5.1)$$

Then, Ғ is a generalized symmetric g-Meir-Keeler type contraction.

Proof. For any $\varepsilon > 0$, it follows from (I) that $\theta(\varepsilon) > 0$. So there exists some α > 0 such that for all u, v, u*, v* ∈ X with gu ≼ gu* and gv ≽ gv*,

$$\theta(\varepsilon) \leq \theta\left(\frac{1}{2}[ȡ(gu, gu^*) + ȡ(gv, gv^*)]\right) < \theta(\varepsilon) + α,$$

implies that $$\theta\left(\frac{1}{2}\left[ȡ\big(Ғ(u, v), Ғ(u^*, v^*)\big) + ȡ\big(Ғ(v, u), Ғ(v^*, u^*)\big)\right]\right) < \theta(\varepsilon). \qquad (4.5.2)$$

By right continuity of θ, there exists some δ > 0 such that $\theta(\varepsilon + \delta) < \theta(\varepsilon) + α$.

For any ϰ, y, u, v ∈ Ⴟ such that gϰ ≼ gu, gy ≽ gv and

$$\varepsilon \leq \frac{1}{2}[ȡ(gϰ, gu) + ȡ(gy, gv)] < \varepsilon + \delta. \qquad (4.5.3)$$

Now, since θ is an increasing function, we can obtain

$$\theta(\varepsilon) \leq \theta\left(\frac{1}{2}[ȡ(gϰ, gu) + ȡ(gy, gv)]\right) < \theta(\varepsilon + \delta) < \theta(\varepsilon) + α. \qquad (4.5.4)$$

Then, by (4.5.2), we get $\theta\left(\frac{1}{2}\left[ȡ\big(Ғ(ϰ, y), Ғ(u, v)\big) + ȡ\big(Ғ(y, ϰ), Ғ(v, u)\big)\right]\right) < \theta(\varepsilon)$ and hence, $\frac{1}{2}\left[ȡ\big(Ғ(ϰ, y), Ғ(u, v)\big) + ȡ\big(Ғ(y, ϰ), Ғ(v, u)\big)\right] < \varepsilon$. Therefore, it follows that Ғ is a generalized symmetric g-Meir-Keeler type contraction. This completes our proof.

The following result is an immediate consequence of Theorems 4.2.1 and 4.5.1:

Corollary 4.5.1. Let (Ҳ, ≼, ḍ) be a POMS and Ғ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be two mappings such that Ғ(Ҳ × Ҳ) ⊆ g(Ҳ), (g(Ҳ), d) is a complete subspace of (Ҳ, ḍ) and the following hypotheses hold:

(IV) Ғ has MSgMP;

(V) for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \int_0^{(1/2)[ḍ(g\varkappa,\, gu) + ḍ(gy,\, gv)]} \phi(f)\, df < \varepsilon + \delta(\varepsilon),$$

implies $$\int_0^{(1/2)\left[ḍ\left(Ғ(\varkappa,y),\, Ғ(u,\, v)\right) + ḍ\left(Ғ(y,\, \varkappa),\, Ғ(v,u)\right)\right]} \phi(f)\, df < \varepsilon, \tag{4.5.5}$$

for all ϰ, y, u, v in Ҳ with gϰ ≼ gu and gy ≽ gv, where ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a locally integrable function satisfying $\int_0^s \phi(f)\, df > 0$ for all s > 0;

(VI) Ҳ has either property (P5) or (P6).

Also, suppose that hypotheses (i) and (ii) of Theorem 4.2.1 hold. Then, Ғ and g have a coupled coincidence point.

Remark 4.5.1. For each $\varepsilon > 0$, taking $\delta(\varepsilon) = (1/ķ - 1)\varepsilon$, $0 < ķ < 1$ in Corollary 4.5.1, we can obtain the coupled coincidence points for Ғ and g under the following contraction (retaining all the other conditions of Corollary 4.5.1):

$$\int_0^{(1/2)[ḍ(g\varkappa,\, gu) + ḍ(gy,\, gv)]} \phi(f)\, df \leq ķ \int_0^{(1/2)\left[ḍ\left(Ғ(\varkappa,y),\, Ғ(u,\, v)\right) + ḍ\left(Ғ(y,\, \varkappa),\, Ғ(v,u)\right)\right]} \phi(f)\, df, \tag{4.5.6}$$

for ϰ, y, u, v in Ҳ with gϰ ≼ gu and gy ≽ gv, where $ķ \in (0, 1)$ and ϕ is a locally integrable function from $\mathbb{R}^+$ into itself satisfying $\int_0^s \phi(f)\, df > 0$ for all s > 0.

FRAMEWORK OF CHAPTER - V

In this chapter, we give some coupled coincidence and coupled common fixed point results in the setup of G-metric spaces with a partial order. Various contractions present in the literature are generalized. Application to the solution of integral equations is also given.

PUBLISHED WORK:

(1) Journal of Inequalities and Applications 2013, **2013**:372.

CONFERENCE:

(1) One paper presented in the International Conference ICSCMM-17 held during Dec 22-23, 2017 at KIET Group of Institutions, Ghaziabad (U.P.) and published in the conference proceedings:
Malaya Journal of Matematik, S(1) (2018), pp. 5-13.
(Some part of this paper is utilized in this chapter and the remaining part is used in Chapter – VI).

CHAPTER – V

COUPLED FIXED POINTS IN G-METRIC SPACES

In this chapter, we study some coupled coincidence and coupled common fixed point results in POGMS. This chapter consists of four sections. Section 5.1 gives a brief introduction to coupled fixed point results in G-metric spaces. In section 5.2, we establish some coupled coincidence and coupled common fixed point theorems for mixed g-monotone mappings satisfying $(\phi, \psi) -$ contractive conditions. Section 5.3 consists of some coupled coincidence and coupled common fixed point results for mappings having MgMP and satisfying new generalized nonlinear contractions. At last, in section 5.4, as application of the obtained results, we discuss the solution of integral equations.

Author's Original Contributions In This Chapter Are:

Theorems: 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.3.1, 5.3.2, 5.4.1.

Corollaries: 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5, 5.3.1.

Examples: 5.2.1, 5.3.1, 5.3.2.

Remarks: 5.2.1, 5.3.1.

5.1 INTRODUCTION

Now-a-days, authors are taking much interest in formulating coupled fixed point results in POGMS. The first coupled fixed point result in POGMS was formulated by Choudhury and Maity [103]. Subsequently, many interesting coupled fixed point results have been developed in G-metric spaces. Below are some definitions and contractions that have been used by different authors to establish their work:

Definition 5.1.1 ([160]). Let Ξ denote the class of functions $\wp$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{(\mathfrak{t}_1,\mathfrak{t}_2)\to(\mathfrak{r}_1,\mathfrak{r}_2)} \wp(\mathfrak{t}_1, \mathfrak{t}_2) > 0$ for all $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\mathfrak{r}_1 + \mathfrak{r}_2 > 0$.

As in the Definition 2.1.13, denote by Φ_1, the class of all functions φ: $\mathbb{R}^+ \to \mathbb{R}^+$ with the properties:

(φ_1) φ is continuous and non-decreasing;

(φ_2) $\varphi(\mathfrak{t}) = 0$ iff $\mathfrak{t} = 0$;

(φ_3) $\varphi(\mathfrak{t} + s) \leq \varphi(\mathfrak{t}) + \varphi(s)$, for all $\mathfrak{t}$, s $\in \mathbb{R}^+$.

As in the Definition 2.1.14, denote by Ψ, the class of all functions ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ with the property: (i_ψ) "$\lim_{\mathfrak{t} \to \mathfrak{r}} \psi(\mathfrak{t}) > 0$ for all $\mathfrak{r} > 0$ and $\lim_{\mathfrak{t}\to 0+} \psi(\mathfrak{t}) = 0$".

Let $(\mathcal{X}, G)$ be a G-metric space and $\preccurlyeq$ be a partial order on $\mathcal{X}$. Let $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g: \mathcal{X} \to \mathcal{X}$ be two mappings. For the mapping F with MMP, Choudhury and Maity [103] gave the following coupled fixed point result:

Theorem 5.1.1 ([103]). Let $(\mathcal{X}, \preccurlyeq, G)$ be a POCGMS and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a continuous mapping having MMP on $\mathcal{X}$. Assume that there exists a $k \in [0, 1)$ such that

$$G(F(x, y), F(u, v), F(w, z) \leq \frac{k}{2}[G(x, u, w) + G(y, v, z)], \tag{5.1.1}$$

for all $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \neq w$ or $v \neq z$. If $\mathcal{X}$ has property **(P1)** which states: "there exist two elements $x_0, y_0 \in \mathcal{X}$ with $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$", then F has a coupled fixed point.

It was also shown in [103] that Theorem 5.1.1 still holds, if the continuity hypothesis of F be replaced by Assumption 2.1.7 w.r.t. convergence and ordering in $(\mathcal{X}, \preccurlyeq, G)$, which is again given below (for convenience):

Assumption 2.1.7 ([55]). X has the property:

(i) "if a non-decreasing sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{X}$ converges to x, then $x_n \preccurlyeq x$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset \mathcal{X}$ converges to y, then $y \preccurlyeq y_n$ for all n".

Nashine [161] obtained coupled coincidence points for a pair of commuting mappings under the following condition:

$$G(F(x, y), F(u, v), F(w, z)) \leq k\,[G(gx, gu, gw) + G(gy, gv, gz)], \tag{5.1.2}$$

for $x, y, u, v, w, z \in \mathcal{X}$ with $gx \succcurlyeq gu \succcurlyeq gw$ and $gy \preccurlyeq gv \preccurlyeq gz$ where either $gu \neq gw$ or $gv \neq gz$ and $k \in [0, \frac{1}{2})$.

On the other hand, Karapinar et. al. [162] generalized the contraction (5.1.1) under the following condition:

$$G(F(x, y), F(u, v), F(w, z)) + G(F(y, x), F(v, u), F(z, w)) \leq k\,[G(gx, gu, gw) + G(gy, gv, gz)], \tag{5.1.3}$$

for $x, y, u, v, w, z \in \mathcal{X}$ with $gx \succcurlyeq gu \succcurlyeq gw$ and $gy \preccurlyeq gv \preccurlyeq gz$ where $k \in [0, 1)$.

Mohiuddine and Alotaibi [163], extended contraction (5.1.1) as follows:

$$\phi(G(F(x, y), F(u, v), F(w, z))) \leq \frac{1}{2}\phi\big(G(x, u, w) + G(y, v, z)\big) - \psi\left(\frac{G(x, u, w) + G(y, v, z)}{2}\right), \tag{5.1.4}$$

for $\varkappa$, y, u, v, w, $\zeta \in X$ with $\varkappa \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq \zeta$ where either $u \neq w$ or $v \neq \zeta$ and $\phi \in \Phi_1$, $\psi \in \Psi$.

Jain and Tas [164] extended the contraction (1.3) in another way by considering the following contraction:

$$\phi\left(\frac{G(F(\varkappa, y), F(u, v), F(w, \zeta)) + G(F(y, \varkappa), F(v, u), F(\zeta, w))}{2}\right)$$
$$\leq \phi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, g\zeta)}{2}\right) - \psi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, g\zeta)}{2}\right), \qquad (5.1.5)$$

for all $\varkappa$, y, ζ, u, v, $w \in X$ with $g\varkappa \succcurlyeq gu \succcurlyeq gw$ and $gy \preccurlyeq gv \preccurlyeq g\zeta$, where $\psi \in \Psi$ and $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and non-decreasing function such that $\phi(f) < f$ for $f > 0$ and $\phi(f + s) \leq \phi(f) + \phi(s)$ for f, $s \in \mathbb{R}^+$.

5.2 COUPLED COMMON FIXED POINTS FOR (ϕ, ψ) – CONTRACTIVE CONDITIONS

In this section, we extend the work of Choudhury and Maity [103] (that is, Theorem 5.1.1) for a pair of commuting mappings in POGMS.

Now, we give our results as follows:

Theorem 5.2.1. Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$, $g: X \to X$ be two mappings. Suppose there exist $\phi \in \Phi_1$ and $\psi \in \Psi$ such that for all $\varkappa$, y, ζ, u, v, $w \in X$ with $gw \preccurlyeq gu \preccurlyeq g\varkappa$ and $gy \preccurlyeq gv \preccurlyeq g\zeta$, we have

$$\phi\left(G(F(\varkappa, y), F(u, v), F(w, \zeta))\right) \leq \frac{1}{2}\phi\big(G(g\varkappa, gu, gw) + G(gy, gv, g\zeta)\big)$$
$$- \psi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, g\zeta)}{2}\right). \qquad (5.2.1)$$

Suppose that both F, g are continuous and commutes, F has the MgMP and $F(X \times X) \subseteq g(X)$. Assume X has the property **(P2)** which states: "there exist two elements $\varkappa_0$, $y_0 \in X$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$". Then, F and g has a coupled coincidence point in X.

Proof. By (P2), there exist $\varkappa_0$, y_0 in X such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. As $F(X \times X) \subseteq g(X)$ and F has MgMP, then as in proof of Theorem 3.2.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in X such that

$$g(\varkappa_{n+1}) = F(\varkappa_n, y_n),\ g(y_{n+1}) = F(y_n, \varkappa_n), \text{ for } n \geq 0, \qquad (5.2.2)$$

and $$g\varkappa_n \preccurlyeq g\varkappa_{n+1},\ gy_n \succcurlyeq gy_{n+1}, \text{ for } n \geq 0. \qquad (5.2.3)$$

Suppose either $g\varkappa_{n+1} = F(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = F(y_n, \varkappa_n) \neq gy_n$, otherwise, the result follows trivially.

As $g\varkappa_n \succcurlyeq g\varkappa_{n-1}$ and $gy_n \preccurlyeq gy_{n-1}$, using (5.2.1) and (5.2.2), we obtain

$$\phi\big(G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n)\big) = \phi\left(G\big(F(\varkappa_n, y_n), F(\varkappa_n, y_n), F(\varkappa_{n-1}, y_{n-1})\big)\right)$$
$$\leq \frac{1}{2}\,\phi\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})\big)$$
$$-\,\psi\left(\frac{G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1})+G(gy_n, gy_n, gy_{n-1})}{2}\right). \qquad (5.2.4)$$

Similarly,

$$\phi\big(G(gy_{n+1}, gy_{n+1}, gy_n)\big) \leq \frac{1}{2}\,\phi\big(G(gy_n, gy_n, gy_{n-1}) + G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1})\big)$$
$$-\,\psi\left(\frac{G(gy_n, gy_n, gy_{n-1}) + G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1})}{2}\right). \qquad (5.2.5)$$

Adding (5.2.4) and (5.2.5), we get

$$\phi\big(G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n)\big) + \phi\big(G(gy_{n+1}, gy_{n+1}, gy_n)\big)$$
$$\leq \phi\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})\big)$$
$$-\,2\,\psi\left(\frac{G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right). \qquad (5.2.6)$$

Using (φ_3), we get

$$\phi\big(G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)\big)$$
$$\leq \phi\big(G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n)\big) + \phi\big(G(gy_{n+1}, gy_{n+1}, gy_n)\big). \qquad (5.2.7)$$

By (5.2.6) and (5.2.7), we obtain

$$\phi\big(G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)\big)$$
$$\leq \phi\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})\big)$$
$$-\,2\psi\left(\frac{G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right) \qquad (5.2.8)$$
$$\leq \phi\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})\big). \qquad (5.2.9)$$

Since ϕ is non-decreasing, using (5.2.9), we have

$$G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)$$
$$\leq G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1}).$$

Denote $\zeta_n = G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)$, so that $\{\zeta_n\}$ is a decreasing sequence. Then, there exists a $\zeta \geq 0$ such that

$$\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}[G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)] = \zeta. \qquad (5.2.10)$$

We claim that $\zeta = 0$. Suppose, on the contrary that $\zeta > 0$. Letting $n \to \infty$ in (5.2.8) and using the properties of ϕ and ψ, we get

$$\phi(\zeta) = \lim_{n\to\infty}\phi(\zeta_n) \leq \lim_{n\to\infty}\left[\phi(\zeta_{n-1}) - 2\psi\left(\frac{\zeta_{n-1}}{2}\right)\right]$$
$$= \phi(\zeta) - 2\lim_{\zeta_{n-1}\to\zeta}\psi\left(\frac{\zeta_{n-1}}{2}\right) < \phi(\zeta),$$

a contradiction. Therefore $\zeta = 0$, so that

$$\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}[G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)] = 0. \quad (5.2.11)$$

We now show $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences.

If possible, let at least one of $\{g\varkappa_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists some $\varepsilon > 0$ for which we can find sub-sequences $\{g\varkappa_{n(k)}\}$, $\{g\varkappa_{m(k)}\}$ of $\{g\varkappa_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$r_k = G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}\big) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \geq \varepsilon. \quad (5.2.12)$$

Also, for m(k), choose n(k) in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (5.2.12). Then, we have

$$G\big(g\varkappa_{n(k)-1}, g\varkappa_{n(k)-1}, g\varkappa_{m(k)}\big) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) < \varepsilon. \quad (5.2.13)$$

Using (5.2.12), (5.2.13) and (G5), we have

$$\begin{aligned}\varepsilon \leq r_k &= G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}\big) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ &\leq \begin{Bmatrix} G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{n(k)-1}\big) + G\big(g\varkappa_{n(k)-1}, g\varkappa_{n(k)-1}, g\varkappa_{m(k)}\big) \\ +G\big(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}\big) + G\big(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}\big) \end{Bmatrix} \\ &< G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{n(k)-1}\big) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + \varepsilon. \quad (5.2.14)\end{aligned}$$

Taking $k \to \infty$ in (5.2.14) and using (5.2.11), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} [G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}\big) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})] = \varepsilon. \quad (5.2.15)$$

Now, using (G5) and the inequality $G(\varkappa, y, y) \leq 2\, G(y, \varkappa, \varkappa)$, we get

$$\begin{aligned}G(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}) &\leq G(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{n(k)+1}) \\ &\quad + G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)}) \\ &\leq 2\, G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{n(k)}) + G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}) \\ &\quad + G(g\varkappa_{m(k)+1}, g\varkappa_{m(k)+1}, g\varkappa_{m(k)}). \quad (5.2.16)\end{aligned}$$

Similarly,

$$\begin{aligned}G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) &\leq 2\, G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{n(k)}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\ &\quad + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)}). \quad (5.2.17)\end{aligned}$$

Adding (5.2.16) and (5.2.17), we get

$$\begin{aligned}r_k &= G\big(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}\big) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ &\leq 2\zeta_{n(k)} + \zeta_{m(k)} + G\big(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}\big) \\ &\quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}).\end{aligned}$$

Since ϕ is non-decreasing and by (φ_3), we get

$$\phi\big(r_k\big) \leq \phi(2\zeta_{n(k)} + \zeta_{m(k)} + G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})$$

$$+ G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}))$$

$$\leq 2\phi(\zeta_{n(k)}) + \phi(\zeta_{m(k)}) + \phi\left(G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})\right)$$

$$+ \phi\left(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\right). \qquad (5.2.18)$$

Also, since n(k) > m(k), $g\varkappa_{n(k)} \succcurlyeq g\varkappa_{m(k)}$ and $gy_{n(k)} \preccurlyeq gy_{m(k)}$, then by (5.2.1) and (5.2.2), we get

$$\phi\left(G(g\varkappa_{n(k)+1}, g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})\right)$$

$$= \phi\left(G\left(F(\varkappa_{n(k)}, y_{n(k)}), F(\varkappa_{n(k)}, y_{n(k)}), F(\varkappa_{m(k)}, y_{m(k)})\right)\right)$$

$$\leq \frac{1}{2}\phi\left(G(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\right)$$

$$- \psi\left(\frac{G(g\varkappa_{n(k)}, g\varkappa_{n(k)}, g\varkappa_{m(k)})+G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})}{2}\right)$$

$$= \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right). \qquad (5.2.19)$$

Similarly,

$$\phi\left(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\right) \leq \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right). \qquad (5.2.20)$$

Using (5.2.18) − (5.2.20), we get

$$\phi(r_k) \leq 2\phi(\zeta_{n(k)}) + \phi(\zeta_{m(k)}) + \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right).$$

Taking $k \to \infty$ in last inequality and using (5.2.11), (5.2.15) and properties of ϕ and ψ, we get

$$\phi(\varepsilon) \leq 2\phi(0) + \phi(0) + \phi(\varepsilon) - 2\lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right) = \phi(\varepsilon) - 2\lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right) < \phi(\varepsilon),$$

a contradiction.

Hence, both $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Now, by completeness of (X, G), there exist $\varkappa$, y in X such that $\{g\varkappa_n\}$ and $\{gy_n\}$ are G-convergent to $\varkappa$ and y, respectively. Then, using Proposition 2.3.3, we get

$$\lim_{n\to\infty} G(g\varkappa_n, g\varkappa_n, \varkappa) = \lim_{n\to\infty} G(g\varkappa_n, \varkappa, \varkappa) = 0, \qquad (5.2.21)$$

$$\lim_{n\to\infty} G(gy_n, gy_n, y) = \lim_{n\to\infty} G(gy_n, y, y) = 0. \qquad (5.2.22)$$

By G-continuity of g and Propositions 2.3.3 and 2.3.4, we have

$$\lim_{n\to\infty} G(gg\varkappa_n, gg\varkappa_n, g\varkappa) = \lim_{n\to\infty} G(gg\varkappa_n, g\varkappa, g\varkappa) = 0, \qquad (5.2.23)$$

$$\lim_{n\to\infty} G(ggy_n, ggy_n, gy) = \lim_{n\to\infty} G(ggy_n, gy, gy) = 0. \qquad (5.2.24)$$

Since $g\varkappa_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$, then by commutativity of F and g, we get

$$gg\varkappa_{n+1} = gF(\varkappa_n, y_n) = F(g\varkappa_n, gy_n), \quad (5.2.25)$$

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, g\varkappa_n). \quad (5.2.26)$$

Since, $\{g\varkappa_n\}$ is G-convergent to $\varkappa$, $\{gy_n\}$ is G-convergent to y and F is G-continuous, then using Definition 2.3.7, the sequence $\{F(g\varkappa_n, gy_n)\}$ is G-convergent to $F(\varkappa, y)$. Now, by uniqueness of limit and using (5.2.23), (5.2.25), we obtain that $F(\varkappa, y) = g\varkappa$. Similarly, we can get $F(y, \varkappa) = gy$. Hence, the result is proved.

Considering g to be the identity mapping in Theorem 5.2.1, we get the following result:

Corollary 5.2.1. Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$ be a mapping. Suppose there exist $\phi \in \Phi_1$ and $\psi \in \Psi$ such that for all $\varkappa, y, z, u, v, w \in X$ with $w \preccurlyeq u \preccurlyeq \varkappa$ and $y \preccurlyeq v \preccurlyeq z$, we have

$$\phi\left(G\big(F(\varkappa, y), F(u, v), F(w, z)\big)\right) \le \frac{1}{2}\,\phi\big(G(\varkappa,u,w) + G(y,v,z)\big) - \psi\left(\frac{G(\varkappa,u,w) + G(y,v,z)}{2}\right). \quad (5.2.27)$$

Suppose F has MMP and is continuous. If X has property (P1), then F has a coupled fixed point in X.

Considering ϕ and g to be the identity mappings in Theorem 5.2.1, we get the following result:

Corollary 5.2.2. Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$ be a mapping. Suppose there exists $\psi \in \Psi$ such that for all $\varkappa, y, z, u, v, w \in X$ with $w \preccurlyeq u \preccurlyeq \varkappa$ and $y \preccurlyeq v \preccurlyeq z$, we have

$$G\big(F(\varkappa, y), F(u, v), F(w, z)\big) \le \frac{G(\varkappa,u,w) + G(y,v,z)}{2} - \psi\left(\frac{G(\varkappa,u,w) + G(y,v,z)}{2}\right). \quad (5.2.28)$$

Suppose F has MMP and is continuous. If X has property (P1), then F has a coupled fixed point in X.

Considering $\phi(t) = t/2$ and $\psi(t) = (1-k)t/2$, $0 \le k < 1$ in Theorem 5.2.1, we get the following result:

Corollary 5.2.3. Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$, $g: X \to X$ be two mappings. Suppose there exists some $k \in [0, 1)$ such that for all $\varkappa, y, z, u, v, w \in X$ with $gw \preccurlyeq gu \preccurlyeq g\varkappa$ and $gy \preccurlyeq gv \preccurlyeq gz$, we have

$$G\big(F(\varkappa, y), F(u, v), F(w, z)\big) \le \frac{k}{2}[G(g\varkappa, gu, gw) + G(gy, gv, gz)]. \quad (5.2.29)$$

Suppose F has MgMP, $F(X \times X) \subseteq g(X)$ and both F, g are continuous and commutes. If X has property (P2), then F and g have a coupled coincidence point in X.

Remark 5.2.1. Corollary 5.2.3 extends Theorem 5.1.1 (Choudhury and Maity [103]) for a pair of commuting mappings. Taking g to be the identity mapping in Corollary 5.2.3, we obtain Theorem 5.1.1.

Next, we replace the continuity assumption of Ғ by considering Assumption 2.1.7 w.r.t. convergence and ordering in POGMS (Ӿ, ≼, G).

Theorem 5.2.2. Let (Ӿ, ≼, G) be a POGMS and Ғ: Ӿ × Ӿ → Ӿ, g: Ӿ → Ӿ be two mappings. Suppose there exist $\phi \in \Phi_1$ and $\psi \in \Psi$ such that (5.2.1) holds for all ϰ, y, ʐ, u, v, w ∈ Ӿ with gw ≼ gu ≼ gϰ and gy ≼ gv ≼ gʐ. Let Ӿ assumes Assumption 2.1.7 w.r.t. convergence and ordering in (Ӿ, ≼, G). Also, let Ғ has MgMP, Ғ(Ӿ × Ӿ) ⊆ g(Ӿ) and g(Ӿ) be G-complete. If Ӿ has property (P2), then Ғ and g have a coupled coincidence point in Ӿ.

Proof. As in proof of Theorem 5.2.1, we can form the G-Cauchy sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ in the G-complete G-metric space g(Ӿ), so there exist some ϰ, y ∈ Ӿ such that $g\varkappa_n \to g\varkappa$ and $gy_n \to gy$ as $n \to \infty$, that is

$$\lim_{n\to\infty} G(g\varkappa_n, g\varkappa, g\varkappa) = \lim_{n\to\infty} G(g\varkappa_n, g\varkappa_n, g\varkappa) = 0, \tag{5.2.30}$$

$$\lim_{n\to\infty} G(gy_n, gy, gy) = \lim_{n\to\infty} G(gy_n, gy_n, gy) = 0. \tag{5.2.31}$$

As $\{g\varkappa_n\}$ is a non-decreasing sequence and $\{gy_n\}$ is a non-increasing sequence, by Assumption 2.1.7 we get $g\varkappa_n \preccurlyeq g\varkappa$ and $gy \preccurlyeq gy_n$ for all $n \geq 0$. Using (5.2.1), we obtain

$$\begin{aligned}\phi\big(G(F(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1})\big) &= \phi\Big(G\big(F(\varkappa,y), F(\varkappa_n,y_n), F(\varkappa_n,y_n)\big)\Big)\\ &\leq \frac{1}{2}\,\phi\big(G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)\big)\\ &\quad - \psi\left(\frac{G(g\varkappa, g\varkappa_n, g\varkappa_n)+G(gy, gy_n, gy_n)}{2}\right).\end{aligned} \tag{5.2.32}$$

Taking $n \to \infty$ in (5.2.32) and using (5.2.30), (5.2.31) and the properties of ϕ and ψ, we get

$$\begin{aligned}\phi\left(\lim_{n\to\infty} G(F(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1})\right) &\leq \frac{1}{2}\phi\left(\lim_{n\to\infty}[G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)]\right)\\ &\quad - \lim_{n\to\infty}\psi\left(\frac{G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)}{2}\right) = 0,\end{aligned}$$

which implies

$$\lim_{n\to\infty} G(F(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1}) = 0. \tag{5.2.33}$$

Also, using (G5), we get

$$G(F(\varkappa,y), g\varkappa, g\varkappa) \leq G(F(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1}) + G(g\varkappa_{n+1}, g\varkappa, g\varkappa).$$

Taking $n \to \infty$ in last inequality and using (5.2.30) and (5.2.33), we get

$$G(Ғ(\varkappa, y), g\varkappa, g\varkappa) = 0, \text{ so that } Ғ(\varkappa, y) = g\varkappa.$$

Similarly, we have $Ғ(y, \varkappa) = gy$. This completes the proof.

Considering ϕ and g to be the identity mappings in Theorem 5.2.2, we get the following result:

Corollary 5.2.4. Let (Ҳ, ≼, G) be a POCGMS and Ғ: Ҳ × Ҳ → Ҳ be a mapping having MMP. Suppose there exists $\psi \in \Psi$ such that (5.2.28) holds for all $\varkappa$, y, ʓ, u, v, w ∈ Ҳ with w ≼ u ≼ $\varkappa$ and y ≼ v ≼ ʓ. Let Ҳ assumes Assumption 2.1.7 w.r.t. convergence and ordering in (Ҳ, ≼, G). If X has property (P1), then Ғ has a coupled fixed point in Ҳ.

Considering $\phi(\mathfrak{f}) = \mathfrak{f}/2$ and $\psi(\mathfrak{f}) = (1-k)\mathfrak{f}/2$, $0 \le k < 1$ in Theorem 5.2.2, we get the following result:

Corollary 5.2.5. Let (Ҳ, ≼, G) be a POGMS, Ғ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be two mappings and and there exists some k ∈ [0, 1) such that (5.2.29) holds for all $\varkappa$, y, ʓ, u, v, w ∈ Ҳ with gw ≼ gu ≼ g$\varkappa$ and gy ≼ gv ≼ gʓ. Let Ҳ assumes Assumption 2.1.7 w.r.t. convergence and ordering in (Ҳ, ≼, G). Further, let Ғ has MgMP, Ғ(Ҳ × Ҳ) ⊆ g(Ҳ) and g(Ҳ) be G-complete. If Ҳ has property (P2), then Ғ and g have a coupled coincidence point in Ҳ.

We now give an example in support of Theorem 5.2.2 as follows:

Example 5.2.1. Consider the POGMS (Ҳ, ≼, G), where Ҳ = [0, 1], the natural ordering ≤ of the real numbers as the partial ordering ≼ and G: Ҳ × Ҳ × Ҳ → $\mathbb{R}^+$ be defined by $G(\varkappa, y, ʓ) = |\varkappa - y| + |y - ʓ| + |ʓ - \varkappa|$ for all $\varkappa$, y, ʓ ∈ Ҳ. Clearly, Ҳ assumes Assumption 2.1.7.

Define the mappings Ғ: Ҳ × Ҳ → Ҳ and g: Ҳ → Ҳ respectively by

$$Ғ(\varkappa, y) = \begin{cases} \frac{\varkappa - y}{24}, & \text{if } \varkappa \ge y, \\ 0, & \text{if } \varkappa < y, \end{cases} \quad \text{and} \quad g\varkappa = \frac{\varkappa}{2} \text{ for all } \varkappa, y \in Ҳ.$$

Then, Ғ has MgMP, g(Ҳ) is G-complete and Ғ(Ҳ × Ҳ) ⊆ g(Ҳ).

Also, define ϕ, ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ respectively by $\phi(\mathfrak{f}) = \frac{\mathfrak{f}}{2}$, $\psi(\mathfrak{f}) = \frac{\mathfrak{f}}{4}$ for $\mathfrak{f} \in \mathbb{R}^+$.

Further, $\varkappa_0 = 0$ and $y_0 = c\ (>0)$ are two points in Ҳ such that $g\varkappa_0 = 0 = Ғ(\varkappa_0, y_0)$ and $gy_0 = \frac{c}{2} \succcurlyeq \frac{c}{24} = Ғ(y_0, \varkappa_0)$.

Now, we verify inequality (5.2.1) for Theorem 5.2.2.

For, taking $\varkappa$, y, ʓ, u, v, w ∈ Ҳ such that g$\varkappa$ ≽ gu ≽ gw and gy ≼ gv ≼ gʓ, so that $\varkappa$ ≥ u ≥ w and y ≤ v ≤ ʓ, we discuss the following cases:

Case 1: $\varkappa \geq y$, $u \geq v$, $w \geq z$.

Then $\phi\left(G\big(Ғ(\varkappa, y), Ғ(u, v), Ғ(w, z)\big)\right) = \phi\left(G\left(\frac{\varkappa - y}{24}, \frac{u - v}{24}, \frac{w - z}{24}\right)\right)$

$= \frac{1}{2}\left\{\frac{|(\varkappa - y) - (u - v)|}{24} + \frac{|(u - v) - (w - z)|}{24} + \frac{|(w - z) - (\varkappa - y)|}{24}\right\}$

$= \frac{1}{48}\{|(\varkappa - u) - (y - v)| + |(u - w) - (v - z)| + |(w - \varkappa) - (z - y)|\}$

$\leq \frac{1}{48}\{(\varkappa - u) + (v - y) + (u - w) + (z - v) + (\varkappa - w) + (z - y)\}$

$= \frac{1}{24}\left\{\left(\frac{\varkappa - u}{2} + \frac{u - w}{2} + \frac{\varkappa - w}{2}\right) + \left(\frac{v - y}{2} + \frac{z - v}{2} + \frac{z - y}{2}\right)\right\}$

$= \frac{1}{24}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\} \leq \frac{1}{8}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\}$

$= \frac{1}{2}\,\phi\big(G(g\varkappa, gu, gw) + G(gy, gv, gz)\big) - \psi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, gz)}{2}\right)$.

Case 2: $\varkappa \geq y$, $u \geq v$, $w < z$.

Then $\phi\left(G\big(Ғ(\varkappa, y), Ғ(u, v), Ғ(w, z)\big)\right) = \phi\left(G\left(\frac{\varkappa - y}{24}, \frac{u - v}{24}, 0\right)\right)$

$= \frac{1}{2}\left\{\frac{|(\varkappa - y) - (u - v)|}{24} + \frac{|(u - v)|}{24} + \frac{|(\varkappa - y)|}{24}\right\}$

$= \frac{1}{48}\{|(\varkappa - u) - (y - v)| + |(u - v)| + |(\varkappa - y)|\}$

$\leq \frac{1}{48}\{(\varkappa - u) + (v - y) + (u - v) + (\varkappa - y)\}$

$= \frac{1}{48}\{(\varkappa - u) + (v - y) + (u - w + w - v) + (\varkappa - w + w - y)\}$

$= \frac{1}{48}\{(\varkappa - u) + (v - y) + (u - w) + (w - v) + (\varkappa - w) + (w - y)\}$

$\leq \frac{1}{48}\{(\varkappa - u) + (v - y) + (u - w) + (z - v) + (\varkappa - w) + (z - y)\}$

$= \frac{1}{24}\left\{\left(\frac{\varkappa - u}{2} + \frac{u - w}{2} + \frac{\varkappa - w}{2}\right) + \left(\frac{v - y}{2} + \frac{z - v}{2} + \frac{z - y}{2}\right)\right\}$

$= \frac{1}{24}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\} \leq \frac{1}{8}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\}$

$= \frac{1}{2}\,\phi\big(G(g\varkappa, gu, gw) + G(gy, gv, gz)\big) - \psi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, gz)}{2}\right)$.

Case 3: $\varkappa \geq y$, $u < v$, $w < z$.

Then $\phi\left(G\big(Ғ(\varkappa, y), Ғ(u, v), Ғ(w, z)\big)\right) = \phi\left(G\left(\frac{\varkappa - y}{24}, 0, 0\right)\right)$

$= \frac{1}{2}\left\{\frac{|(\varkappa - y)|}{24} + \frac{|(\varkappa - y)|}{24}\right\}$

$= \frac{1}{2}\left\{\frac{(\varkappa - y)}{24} + \frac{(\varkappa - y)}{24}\right\}$

$= \frac{1}{48}\{(\varkappa - u + u - y) + (\varkappa - w + w - y)\}$

$= \frac{1}{48}\{(\varkappa - u) + (u - y) + (\varkappa - w) + (w - y)\}$

$\leq \frac{1}{48}\{(\varkappa - u) + (v - y) + (\varkappa - w) + (w - u + u - y)\}$

$= \frac{1}{48}\{(\varkappa - u) + (v - y) + (\varkappa - w) + (w - u) + (u - y)\}$

$= \frac{1}{48}\{(\varkappa - u) + (v - y) + (\varkappa - w) + (w - u) + (u - z + z - y)\}$

$\leq \frac{1}{48}\{(\varkappa - u) + (v - y) + (\varkappa - w) + (u - w) + (v - z) + (z - y)\}$

$\leq \frac{1}{24}\left\{\left(\frac{\varkappa - u}{2} + \frac{u - w}{2} + \frac{\varkappa - w}{2}\right) + \left(\frac{v - y}{2} + \frac{z - v}{2} + \frac{z - y}{2}\right)\right\}$

$= \frac{1}{24}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\} \leq \frac{1}{8}\{G(g\varkappa, gu, gw) + G(gy, gv, gz)\}$

$= \frac{1}{2}\,\phi\big(G(g\varkappa, gu, gw) + G(gy, gv, gz)\big) - \psi\left(\frac{G(g\varkappa, gu, gw) + G(gy, gv, gz)}{2}\right).$

Case 4: $\varkappa < y$, $u < v$, $w < z$.

Then, $\phi\left(G\big(F(\varkappa, y), F(u, v), F(w, z)\big)\right) = \phi(0) = 0$ so that the inequality (5.2.1) is obvious for Theorem 5.2.2.

Similarly, the cases like $\varkappa < y$, $u \geq v$, $w \geq z$; $\varkappa < y$, $u < v$, $w \geq z$ and others follow immediately. Therefore, all the conditions of Theorem 5.2.2 hold and hence, (0, 0) is the coupled coincidence point of F and g.

Coupled Common Fixed Points

We now establish the existence and uniqueness of common coupled fixed points. For, we require Assumption 3.2.1, again given below (for convenience):

Assumption 3.2.1 ([59]). "For every $(\varkappa, y), (\varkappa^*, y^*) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(\varkappa, y), F(y, x))$ and $(F(\varkappa^*, y^*), F(y^*, \varkappa^*))$".

Theorem 5.2.3. In addition to the hypotheses of Theorem 5.2.1, if Assumption 3.2.1 holds, then, F and g have a unique coupled common fixed point in X.

Proof. By Theorem 5.2.1, the set of coupled coincidences is non-empty. To prove the result, we first show that if $(\varkappa, y)$ and $(\varkappa^*, y^*)$ are coupled coincidence points of F and g, then

$$g\varkappa = g\varkappa^* \text{ and } gy = gy^*. \qquad (5.2.34)$$

By Assumption 3.2.1, there is some $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(\varkappa, y), F(y, \varkappa))$ and $(F(\varkappa^*, y^*), F(y^*, \varkappa^*))$. Taking $u_0 = u$ and $v_0 = v$ and choosing $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$, $gv_1 = F(v_0, u_0)$. Now, as in proof of Theorem 5.2.1, we inductively define the sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$. Further, taking $\varkappa_0 = \varkappa$, $y_0 = y$, $\varkappa_0^* = \varkappa^*$, $y_0^* = y^*$ and on the same way, we define the sequences $\{g\varkappa_n\}$, $\{gy_n\}$, $\{g\varkappa_n^*\}$ and $\{gy_n^*\}$. Then, it is easy to obtain that $g\varkappa_{n+1} = Ғ(\varkappa_n, y_n)$, $gy_{n+1} = Ғ(y_n, \varkappa_n)$ and $g\varkappa^*_{n+1} = Ғ(\varkappa^*_n, y^*_n)$, $gy^*_{n+1} = Ғ(y^*_n, \varkappa^*_n)$ for all $n \geq 0$. Since $(Ғ(\varkappa, y), Ғ(y, \varkappa)) = (g\varkappa_1, gy_1) = (g\varkappa, gy)$ and $(Ғ(u, v), Ғ(v, u)) = (gu_1, gv_1)$ are comparable, then $g\varkappa \preccurlyeq gu_1$ and $gy \succcurlyeq gv_1$. It is easy to obtain that $(g\varkappa, gy)$ and (gu_n, gv_n) are comparable, that is $g\varkappa \preccurlyeq gu_n$ and $gy \succcurlyeq gv_n$ for all $n \geq 1$. Then by (5.2.1), we get

$$\phi\big(G(gu_{n+1}, gu_{n+1}, g\varkappa)\big) = \phi\Big(G\big(Ғ(u_n, v_n), Ғ(u_n, v_n), Ғ(\varkappa, y)\big)\Big)$$
$$\leq \frac{1}{2}\phi\big(G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)\big)$$
$$- \psi\left(\frac{G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)}{2}\right) \qquad (5.2.35)$$

and $$\phi\big(G(gv_{n+1}, gv_{n+1}, gy)\big) \leq \frac{1}{2}\phi\big(G(gv_n, gv_n, gy) + G(gu_n, gu_n, g\varkappa)\big)$$
$$- \psi\left(\frac{G(gv_n, gv_n, gy) + G(gu_n, gu_n, g\varkappa)}{2}\right). \qquad (5.2.36)$$

Adding (5.2.35) and (5.2.36), we have

$$\phi\big(G(gu_{n+1}, gu_{n+1}, g\varkappa)\big) + \phi\big(G(gv_{n+1}, gv_{n+1}, gy)\big)$$
$$\leq \phi\big(G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)\big) - 2\psi\left(\frac{G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)}{2}\right).$$
$$(5.2.37)$$

Also, by (φ_3) we have

$$\phi\big(G(gu_{n+1}, gu_{n+1}, g\varkappa) + G(gv_{n+1}, gv_{n+1}, gy)\big)$$
$$\leq \phi\big(G(gu_{n+1}, gu_{n+1}, g\varkappa)\big) + \phi\big(G(gv_{n+1}, gv_{n+1}, gy)\big). \qquad (5.2.38)$$

By (5.2.37) and (5.2.38), we get

$$\phi\big(G(gu_{n+1}, gu_{n+1}, g\varkappa) + G(gv_{n+1}, gv_{n+1}, gy)\big)$$
$$\leq \phi\big(G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)\big) - 2\,\psi\left(\frac{G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)}{2}\right)$$
$$(5.2.39)$$
$$\leq \phi\big(G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)\big). \qquad (5.2.40)$$

As ϕ is non-decreasing function, by (5.2.40), we have

$G(gu_{n+1}, gu_{n+1}, g\varkappa) + G(gv_{n+1}, gv_{n+1}, gy) \leq G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)$.

Denote $б_n = G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)$, then $\{б_n\}$ is a non-increasing sequence, so there exists some $б \geq 0$ such that $\lim_{n\to\infty} б_n = б$. We claim $б = 0$. On the contrary, let $б > 0$. Now, taking $n \to \infty$ in (5.2.39) and using the continuity of ϕ and the property (i_ψ), we get $\phi(б) \leq \phi(б) - 2\lim_{б_n \to б} \psi\left(\frac{б_n}{2}\right) < \phi(б)$, a contradiction. Thus, $б = 0$, so that $\lim_{n\to\infty} б_n = \lim_{n\to\infty}[G(gu_n, gu_n, g\varkappa) + G(gv_n, gv_n, gy)] = 0$.

Therefore, we get $gu_n \to g\varkappa$, $gv_n \to gy$. Similarly, we can obtain that $gu_n \to g\varkappa^*$, $gv_n \to gy^*$. Now, by uniqueness of limit, we get $g\varkappa = g\varkappa^*$ and $gy = gy^*$. Therefore, we have proved (5.2.34). Since $g\varkappa = Ғ(\varkappa, y)$, $gy = Ғ(y, \varkappa)$ and the pair $(Ғ, g)$ is commuting, we have

$$gg\varkappa = gҒ(\varkappa, y) = Ғ(g\varkappa, gy) \text{ and } ggy = gҒ(y, \varkappa) = Ғ(gy, g\varkappa). \tag{5.2.41}$$

Denote $g\varkappa = ʐ$, $gy = ⱳ$. Then by (5.2.41), we get

$$gʐ = Ғ(ʐ, ⱳ) \text{ and } gⱳ = Ғ(ⱳ, ʐ). \tag{5.2.42}$$

Therefore, $(ʐ, ⱳ)$ is a coupled coincidence point of $Ғ$ and g. Now, using (5.2.34) for $\varkappa^* = ʐ$ and $y^* = ⱳ$, we have $gʐ = g\varkappa$ and $gⱳ = gy$, that is

$$gʐ = ʐ,\ gⱳ = ⱳ. \tag{5.2.43}$$

Now, using (5.2.42) and (5.2.43), we have $ʐ = gʐ = Ғ(ʐ, ⱳ)$ and $ⱳ = gⱳ = Ғ(ⱳ, ʐ)$, so that $(ʐ, ⱳ)$ is a coupled common fixed point of $Ғ$ and g. Now, for uniqueness, suppose (s, r) be a coupled common fixed point of $Ғ$ and g. Then by (5.2.34), we get $s = gs = gʐ = ʐ$ and $r = gr = gⱳ = ⱳ$.

Theorem 5.2.4. In addition to the hypotheses of Theorem 5.2.2 suppose Assumption 3.2.1 also holds. If $Ғ$ and g commutes, then they have a unique coupled common fixed point in $Ꭓ$.

Proof. Following the steps of Theorem 5.2.3, the proof follows immediately.

5.3 COUPLED COMMON FIXED POINTS UNDER NEW NONLINEAR CONTRACTION

In this section, we generalize the contractions involved in the works of Karapinar et al. [162], Jain and Tas [164] (that is, contractions (5.1.3) and (5.1.5), respectively) and weaken the contractions involved in results of Choudhury and Maity [103], Nashine [161] and Mohiuddine and Alotaibi [163] (that is, contractions (5.1.1), (5.1.2) and (5.1.4), respectively).

Theorem 5.3.1. Let $(Ꭓ, \preccurlyeq, G)$ be a POCGMS and $Ғ: Ꭓ \times Ꭓ \to Ꭓ$, $g: Ꭓ \to Ꭓ$ be the mappings. Suppose there exist some $\wp \in \Xi$ and an ADF π such that for all l, m, n, u, v, $ⱳ \in Ꭓ$ with $gⱳ \preccurlyeq gu \preccurlyeq gl$ and $gm \preccurlyeq gv \preccurlyeq gn$, we have

$$\pi\left(\frac{G(Ғ(l,m), Ғ(u,v), Ғ(ⱳ,n)) + G(Ғ(m,l), Ғ(v,u), Ғ(n, ⱳ))}{2}\right)$$
$$\leq \pi\left(\frac{G(gl, gu, gⱳ) + G(gm, gv, gn)}{2}\right) - \wp\big(G(gl, gu, gⱳ), G(gm, gv, gn)\big). \tag{5.3.1}$$

Assume $Ғ(Ꭓ \times Ꭓ) \subseteq g(Ꭓ)$, $Ғ$ has MgMP and $Ғ$, g are both continuous and commutes. If $Ꭓ$ has the property (P2), then, $Ғ$ and g have a coupled coincidence point in $Ꭓ$.

Proof. By (P2), there exist $\varkappa_0, y_0 \in \mathfrak{X}$ with $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. As $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$ and F has MgMP, then as in proof of Theorem 3.2.1, we construct sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ in $\mathfrak{X}$ such that

$$g\varkappa_n = F(\varkappa_{n-1}, y_{n-1}) \text{ and } gy_n = F(y_{n-1}, \varkappa_{n-1}), \text{ for all } n \geq 0, \tag{5.3.2}$$

and

$$g\varkappa_n \preccurlyeq g\varkappa_{n+1} \text{ and } gy_n \succcurlyeq gy_{n+1}. \tag{5.3.3}$$

We suppose that $(g\varkappa_{n+1}, gy_{n+1}) \neq (g\varkappa_n, gy_n)$ for all $n \in \mathbb{N}$, otherwise, we obtain the result directly.

As $g\varkappa_n \succcurlyeq g\varkappa_{n-1}$ and $gy_n \preccurlyeq gy_{n-1}$, then using (5.3.1) and (5.3.2), we get

$$\begin{aligned}
&\pi\left(\frac{G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)}{2}\right) \\
&\quad = \pi\left(\frac{G\big(F(\varkappa_n, y_n), F(\varkappa_n, y_n), F(\varkappa_{n-1}, y_{n-1})\big) + G\big(F(y_n, \varkappa_n), F(y_n, \varkappa_n), F(y_{n-1}, \varkappa_{n-1})\big)}{2}\right) \\
&\quad \leq \pi\left(\frac{G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right) \\
&\qquad - \wp\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}), G(gy_n, gy_n, gy_{n-1})\big).
\end{aligned} \tag{5.3.4}$$

Since $\wp(\mathfrak{t}_1, \mathfrak{t}_2) \geq 0$, for all $(\mathfrak{t}_1, \mathfrak{t}_2) \in (\mathbb{R}^+)^2$, we have

$$\pi\left(\frac{G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)}{2}\right) \leq \pi\left(\frac{G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right), \tag{5.3.5}$$

which implies on using the monotone property of π that $\{\zeta_n\}$ is a non-increasing sequence, where $\zeta_n = G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)$ and hence, there exists some $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} [G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) + G(gy_{n+1}, gy_{n+1}, gy_n)] = \zeta. \tag{5.3.6}$$

We claim that $\zeta = 0$. On the contrary, suppose that $\zeta > 0$.

Using (5.3.6), the sequences $\{G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n)\}$ and $\{G(gy_{n+1}, gy_{n+1}, gy_n)\}$ have convergent sub-sequences that we also denote by $\{G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n)\}$ and $\{G(gy_{n+1}, gy_{n+1}, gy_n)\}$, respectively. Let $\lim_{n\to\infty} G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) = \zeta_1$ and $\lim_{n\to\infty} G(gy_{n+1}, gy_{n+1}, gy_n) = \zeta_2$, then $\zeta_1 + \zeta_2 = \zeta > 0$.

On letting $n \to \infty$ in (5.2.4), then using (5.3.6), the continuity of π and the property of $\wp$, we get

$$\pi\left(\frac{\zeta}{2}\right) \leq \pi\left(\frac{\zeta}{2}\right) - \lim_{n\to\infty} \wp\big(G(g\varkappa_n, g\varkappa_n, g\varkappa_{n-1}), G(gy_n, gy_n, gy_{n-1})\big) < \pi\left(\frac{\zeta}{2}\right),$$

a contradiction. Therefore $\zeta = 0$, so that

$$\lim_{n\to\infty} G(g\varkappa_{n+1}, g\varkappa_{n+1}, g\varkappa_n) = 0, \tag{5.3.7}$$

and

$$\lim_{n\to\infty} G(gy_{n+1}, gy_{n+1}, gy_n) = 0. \tag{5.3.8}$$

We now show that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. On the contrary, let at least one of $\{g\varkappa_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. So, there exists an $\varepsilon > 0$ and sequences $\{m(j)\}$ and $\{k(j)\}$ of natural numbers such that for all natural numbers j, $k(j) > m(j) > j$, we have

$$a_j = G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}\big) + G\big(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}\big) \geq \varepsilon, \tag{5.3.9}$$

and

$$G\big(g\varkappa_{k(j)-1}, g\varkappa_{k(j)-1}, g\varkappa_{m(j)}\big) + G\big(gy_{k(j)-1}, gy_{k(j)-1}, gy_{m(j)}\big) < \varepsilon. \tag{5.3.10}$$

Using (G5), we get

$$G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}\big) \leq G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{k(j)-1}\big) + G\big(g\varkappa_{k(j)-1}, g\varkappa_{k(j)-1}, g\varkappa_{m(j)}\big) \tag{5.3.11}$$

and

$$G\big(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}\big) \leq G\big(gy_{k(j)}, gy_{k(j)}, gy_{k(j)-1}\big) + G\big(gy_{k(j)-1}, gy_{k(j)-1}, gy_{m(j)}\big). \tag{5.3.12}$$

Using (5.3.9) – (5.3.12), we get

$$\begin{aligned}\varepsilon \leq a_j &= G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}\big) + G\big(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}\big)\\ &\leq G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{k(j)-1}\big) + G\big(g\varkappa_{k(j)-1}, g\varkappa_{k(j)-1}, g\varkappa_{m(j)}\big)\\ &\quad + G\big(gy_{k(j)}, gy_{k(j)}, gy_{k(j)-1}\big) + G\big(gy_{k(j)-1}, gy_{k(j)-1}, gy_{m(j)}\big).\\ &< G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{k(j)-1}\big) + G\big(gy_{k(j)}, gy_{k(j)}, gy_{k(j)-1}\big) + \varepsilon.\end{aligned}$$

Taking $j \to \infty$ in the last inequality and using (5.3.7) and (5.3.8), we have

$$\lim_{j\to\infty}\big[G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}\big) + G\big(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}\big)\big] = \lim_{j\to\infty} a_j = \varepsilon. \tag{5.3.13}$$

Since $G(a, a, b) \leq 2\,G(a, b, b)$ for any $a, b \in X$, using properties (G2) − (G4), we obtain that

$$\begin{aligned}G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}\big) &\leq G\big(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{k(j)+1}\big) + G\big(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)}\big)\\ &\leq 2\,G\big(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{k(j)}\big) + G\big(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)+1}\big)\\ &\quad + G\big(g\varkappa_{m(j)+1}, g\varkappa_{m(j)+1}, g\varkappa_{m(j)}\big).\end{aligned} \tag{5.3.14}$$

Similarly,

$$\begin{aligned}G\big(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}\big) &\leq 2\,G\big(gy_{k(j)+1}, gy_{k(j)+1}, gy_{k(j)}\big)\\ &\quad + G\big(gy_{k(j)+1}, gy_{k(j)+1}, gy_{m(j)+1}\big) + G\big(gy_{m(j)+1}, gy_{m(j)+1}, gy_{m(j)}\big).\end{aligned} \tag{5.3.15}$$

Using (5.3.14) and (5.3.15), we get

$$\mathfrak{a}_j = G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}) + G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})$$
$$\leq 2\zeta_{k(j)} + \zeta_{m(j)} + G(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1}, gy_{m(j)+1}). \tag{5.3.16}$$

By the properties of π, we get

$$\pi\left(\frac{\mathfrak{a}_j}{2}\right) \leq \pi\left(\zeta_{k(j)} + \frac{\zeta_{m(j)}}{2} + \frac{G(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1}, gy_{m(j)+1})}{2}\right). \tag{5.3.17}$$

Taking j → ∞ in (5.3.17) and using (5.3.7), (5.3.8), (5.3.13) and the continuity of π, we get

$$\pi\left(\frac{\varepsilon}{2}\right) \leq \lim_{j\to\infty} \pi\left(\frac{G(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1}, gy_{m(j)+1})}{2}\right). \tag{5.3.18}$$

Since k(j) > m(j), $g\varkappa_{k(j)} \succcurlyeq g\varkappa_{m(j)}$ and $gy_{k(j)} \preccurlyeq gy_{m(j)}$, by (5.3.1), we have

$$\pi\left(\frac{G(g\varkappa_{k(j)+1}, g\varkappa_{k(j)+1}, g\varkappa_{m(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1}, gy_{m(j)+1})}{2}\right)$$
$$= \pi\left(\frac{G(F(\varkappa_{k(j)}, y_{k(j)}), F(\varkappa_{k(j)}, y_{k(j)}), F(\varkappa_{m(j)}, y_{m(j)})) + G(F(y_{k(j)}, \varkappa_{k(j)}), F(y_{k(j)}, \varkappa_{k(j)}), F(y_{m(j)}, \varkappa_{m(j)}))}{2}\right)$$
$$\leq \pi\left(\frac{G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}) + G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})}{2}\right)$$
$$- \wp\left(G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}), G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})\right)$$
$$= \pi\left(\frac{\mathfrak{a}_j}{2}\right) - \wp\left(G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}), G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})\right). \tag{5.3.19}$$

Using (5.3.18) and (5.3.19), we get

$$\pi\left(\frac{\varepsilon}{2}\right) \leq \lim_{j\to\infty}\left[\pi\left(\frac{\mathfrak{a}_j}{2}\right) - \wp\left(G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}), G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})\right)\right]. \tag{5.3.20}$$

By (5.3.13), the sequences $\{G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)})\}$, $\{G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})\}$ have sub-sequences converging to say, ε_1 and ε_2 respectively and $\varepsilon_1 + \varepsilon_2 = \varepsilon > 0$. Now, passing to the sub-sequences, we suppose that

$$\lim_{j\to\infty} G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}) = \varepsilon_1 \text{ and } \lim_{j\to\infty} G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)}) = \varepsilon_2.$$

Using (5.3.13) and the properties of π, $\wp$ in (5.3.20), we get

$$\pi\left(\frac{\varepsilon}{2}\right) \leq \pi\left(\frac{\varepsilon}{2}\right) - \lim_{j\to\infty} \wp\left(G(g\varkappa_{k(j)}, g\varkappa_{k(j)}, g\varkappa_{m(j)}), G(gy_{k(j)}, gy_{k(j)}, gy_{m(j)})\right)$$
$$< \pi\left(\frac{\varepsilon}{2}\right), \text{ a contradiction.}$$

Hence, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences in the complete G-metric space $(\mathfrak{X}, G)$, so there exist some $\varkappa$, $y \in \mathfrak{X}$ for which $\{g\varkappa_n\}$ is G-convergent to $\varkappa$ and $\{gy_n\}$ is G-convergent to y, then by Proposition 2.3.3, we obtain

$$\lim_{n\to\infty} G(g\varkappa_n, g\varkappa_n, \varkappa) = \lim_{n\to\infty} G(g\varkappa_n, \varkappa, \varkappa) = 0 \tag{5.3.21}$$

and
$$\lim_{n\to\infty} G(gy_n, gy_n, y) = \lim_{n\to\infty} G(gy_n, y, y) = 0. \tag{5.3.22}$$

By continuity of g and Proposition 2.3.4, we have

$$\lim_{n\to\infty} G(gg\varkappa_n, gg\varkappa_n, g\varkappa) = \lim_{n\to\infty} G(gg\varkappa_n, g\varkappa, g\varkappa) = 0, \tag{5.3.23}$$

$$\lim_{n\to\infty} G(ggy_n, ggy_n, gy) = \lim_{n\to\infty} G(ggy_n, gy, gy) = 0. \tag{5.3.24}$$

As $g\varkappa_{n+1} = F(\varkappa_n, y_n)$, $gy_{n+1} = F(y_n, \varkappa_n)$ and mappings F and g commutes, we get

$$gg\varkappa_{n+1} = gF(\varkappa_n, y_n) = F(g\varkappa_n, gy_n), \tag{5.3.25}$$

$$ggy_{n+1} = gF(y_n, \varkappa_n) = F(gy_n, g\varkappa_n). \tag{5.3.26}$$

We next claim that $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

As F is continuous and $\{g\varkappa_n\}$, $\{gy_n\}$ are G-convergent to $\varkappa$, y respectively, then, using Definition 2.3.7, we get that $\{F(g\varkappa_n, gy_n)\}$ is G-convergent to $F(\varkappa, y)$. Hence, using (5.3.25), $\{gg\varkappa_{n+1}\}$ is G-convergent to $F(\varkappa, y)$. Now, using (5.2.23) and the uniqueness of limit, we get $F(\varkappa, y) = g\varkappa$. Similarly, $F(y, \varkappa) = gy$. Therefore, F and g have a coupled coincidence point in $\mathfrak{X}$.

The following example illustrates that contraction (5.3.1) is more general than contraction (5.1.2) (due to Nashine [161]).

Example 5.3.1. Consider the POCGMS $(\mathfrak{X}, \preccurlyeq, G)$ where $\mathfrak{X} = \mathbb{R}$, the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$ and $G: \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ be defined by $G(l, m, n) = |l - m| + |m - n| + |n - l|$ for all $l, m, n \in \mathfrak{X}$. Define the mappings $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ respectively by $F(\varkappa, y) = \frac{\varkappa - 2y}{12}$ and $g\varkappa = \frac{\varkappa}{3}$, for $\varkappa, y \in \mathfrak{X}$. Clearly, F and g both are continuous and commutes, F has MgMP and $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$. Further, we claim that (5.3.1) holds but (5.1.2) does not hold.

Suppose that there exists some $k \in [0, \frac{1}{2})$ such that (5.1.2) holds. Then, we shall have

$$\left(\left|\frac{l-2m}{12} - \frac{u-2v}{12}\right| + \left|\frac{u-2v}{12} - \frac{w-2n}{12}\right| + \left|\frac{w-2n}{12} - \frac{l-2m}{12}\right|\right)$$
$$\leq k\left[\left(\left|\frac{l}{3} - \frac{u}{3}\right| + \left|\frac{u}{3} - \frac{w}{3}\right| + \left|\frac{w}{3} - \frac{l}{3}\right|\right) + \left(\left|\frac{m}{3} - \frac{v}{3}\right| + \left|\frac{v}{3} - \frac{n}{3}\right| + \left|\frac{n}{3} - \frac{m}{3}\right|\right)\right]$$
$$= \frac{k}{3}[(|l - u| + |u - w| + |w - l|) + (|m - v| + |v - n| + |n - m|)]$$

for all $l \geq u \geq w$ and $m \leq v \leq n$. Take $l = u = w$, $v \neq n$, $m = v$, $m \neq n$ in last inequality and $\rho = |n - v| + |n - m|$, we get $\frac{\rho}{2} \leq k\rho$, $\rho > 0$, which implies $\frac{1}{2} \leq k$, a contradiction since $k \in [0, \frac{1}{2})$. Therefore, (5.1.2) does not hold.

Now, we show that (5.3.1) holds.

For $l \geq u \geq w$ and $m \leq v \leq n$, we have

$$\left|\frac{l-2m}{12} - \frac{u-2v}{12}\right| \leq \frac{1}{12}|l - u| + \frac{1}{6}|m - v|, \quad \left|\frac{u-2v}{12} - \frac{w-2n}{12}\right| \leq \frac{1}{12}|u - w| + \frac{1}{6}|v - n|,$$

$$\left|\frac{w-2n}{12} - \frac{l-2m}{12}\right| \leq \frac{1}{12}|w - l| + \frac{1}{6}|n - m|, \quad \left|\frac{m-2l}{12} - \frac{v-2u}{12}\right| \leq \frac{1}{12}|m - v| + \frac{1}{6}|l - u|,$$

$$\left|\frac{v-2u}{12} - \frac{n-2w}{12}\right| \leq \frac{1}{12}|v - n| + \frac{1}{6}|u - w|, \quad \left|\frac{n-2w}{12} - \frac{m-2l}{12}\right| \leq \frac{1}{12}|n - m| + \frac{1}{6}|w - l|.$$

Adding these six inequalities, we obtain (5.3.1) for $\pi(t) = \frac{t}{3}$, $\wp(t_1, t_2) = \frac{(t_1+t_2)}{12}$. Further, take $\varkappa_0 = -2$, $y_0 = 2$ are the elements of X so that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$, $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Now, all the conditions of Theorem 5.3.1 holds. By Theorem 5.3.1, F and g have a coupled coincidence point (0, 0) in X.

Considering g to be the identity mapping in Theorem 5.3.1, we get the following result:

Corollary 5.3.1. Let $(X, \preccurlyeq, G)$ be a POCGMS and $F: X \times X \to X$ be a continuous mapping with MMP. Suppose there exist some $\wp \in \Xi$ and an ADF π such that for all $l, m, n, u, v, w \in X$ with $gw \preccurlyeq gu \preccurlyeq gl$ and $gm \preccurlyeq gv \preccurlyeq gn$, we have

$$\pi\left(\frac{G\big(F(l,m), F(u,v), F(w,n)\big) + G\big(F(m,l), F(v,u), F(n,w)\big)}{2}\right)$$
$$\leq \pi\left(\frac{G(l,u,w) + G(m,v,n)}{2}\right) - \wp\big(G(l,u,w), G(m,v,n)\big). \quad (5.3.27)$$

If X has the property (P1), then, F has a coupled fixed point in X.

Remark 5.3.1. (i) In Theorem 5.3.1, taking $\wp(t_1, t_2) = \psi\left(\frac{t_1+t_2}{2}\right)$ for $t_1, t_2 \in \mathbb{R}^+$ with $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ staisfying "$\lim_{t \to r} \psi(t) > 0$ for each $r > 0$", the contraction (5.3.1) becomes

$$\pi\left(\frac{G\big(F(l,m), F(u,v), F(w,n)\big) + G\big(F(m,l), F(v,u), F(n,w)\big)}{2}\right)$$
$$\leq \pi\left(\frac{G(gl, gu, gw) + G(gm, gv, gn)}{2}\right) - \psi\left(\frac{G(gl, gu, gw) + G(gm, gv, gn)}{2}\right), \quad (5.3.28)$$

which is analogous to contraction (5.1.5) (due to Jain and Tas [164]).

(ii) In Theorem 5.3.1, taking π to be the identity mapping and $\wp(t_1, t_2) = \frac{1-k}{2}(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}^+$ with $k \in [0, 1)$, the contraction (5.3.1) becomes

$$G\big(F(l, m), F(u, v), F(w, n)\big) + G\big(F(m, l), F(v, u), F(n, w)\big)$$

$$\leq \Bbbk[G(gl, gu, gw) + G(gm, gv, gn)], \qquad (5.3.29)$$

which is contraction (5.1.3) (due to Karapinar et al. [162]).

The next example furnishes the fact that contraction (5.3.27) is more general than the contractions (5.1.1) (due to Choudhury and Maity [103]) and (5.1.4) (due to Mohiuddine and Alotaibi [163]).

Example 5.3.2. Consider the POCGMS (Ҳ, ≼, G) with Ҳ = ℝ, the natural ordering ≤ of the real numbers as the partial ordering ≼ and G: Ҳ × Ҳ × Ҳ → $\mathbb{R}^+$ be defined by G(l, m, n) = $|l - m| + |m - n| + |n - l|$ for all l, m, n ∈ Ҳ.

Define the mapping Ғ: Ҳ × Ҳ → Ҳ by Ғ(ϰ, y) = $\frac{\varkappa - 4y}{8}$ for ϰ, y ∈ Ҳ.

Then, Ғ is continuous and has MMP. We claim that here (5.3.27) holds but (5.1.1) and (5.1.4) do not hold.

Let there exists some ķ ∈ [0, 1) such that (5.1.1) holds, then, we have

$$\left|\frac{l-4m}{8} - \frac{u-4v}{8}\right| + \left|\frac{u-4v}{8} - \frac{w-4n}{8}\right| + \left|\frac{w-4n}{8} - \frac{l-4m}{8}\right|$$
$$\leq \frac{k}{2}\{(|l - u| + |u - w| + |w - l|) + (|m - v| + |v - n| + |n - m|)\},$$

for $l \geq u \geq w$ and $m \leq v \leq n$. Take $l = u = w$, $v \neq n$, $m = v$, $m \neq n$ in last inequality and $\rho = |n - v| + |n - m|$, we get $\rho \leq k\,\rho$, $\rho > 0$, which implies $1 \leq k$, a contradiction, since ķ ∈ [0, 1). Therefore, (5.1.1) does not hold.

Now, if (5.1.4) holds for some ϕ and ψ, then, for $l \geq u \geq w$ and $m \leq v \leq n$, we shall have

$$\phi\left(\left|\frac{l-4m}{8} - \frac{u-4v}{8}\right| + \left|\frac{u-4v}{8} - \frac{w-4n}{8}\right| + \left|\frac{w-4n}{8} - \frac{l-4m}{8}\right|\right)$$
$$\leq \frac{1}{2}\phi(|l - u| + |u - w| + |w - l| + |m - v| + |v - n| + |n - m|)$$
$$- \psi\left(\frac{|l-u|+|u-w|+|w-l|+|m-v|+|v-n|+|n-m|}{2}\right),$$

by which for $l = u = w$, $v \neq n$, $m = v$, $m \neq n$, we have

$$\phi\left(\frac{1}{2}(|n - v| + |n - m|)\right) \leq \frac{1}{2}\phi(|n - v| + |n - m|) - \psi\left(\frac{1}{2}(|n - v| + |n - m|)\right),$$

then, for $\rho = \frac{1}{2}(|n - v| + |n - m|)$, using the last inequality we get

$$\phi(\rho) \leq \frac{1}{2}\phi(2\rho) - \psi(\rho) \leq \phi(\rho) - \psi(\rho) \qquad \text{(by property of } \phi\text{)}$$
$$< \phi(\rho), \text{ a contradiction.}$$

We finally show that (5.3.27) holds. For $l \geq u \geq w$ and $m \leq v \leq n$, we have

$$\left|\frac{l-4m}{8} - \frac{u-4v}{8}\right| \leq \frac{1}{8}|l - u| + \frac{1}{2}|m - v|, \quad \left|\frac{u-4v}{8} - \frac{w-4n}{8}\right| \leq \frac{1}{8}|u - w| + \frac{1}{2}|v - n|,$$

$\left|\frac{w-4n}{8}-\frac{l-4m}{8}\right| \leq \frac{1}{8}|w-l|+\frac{1}{2}|n-m|, \quad \left|\frac{m-4l}{8}-\frac{v-4u}{8}\right| \leq \frac{1}{8}|m-v|+\frac{1}{2}|l-u|,$

$\left|\frac{v-4u}{8}-\frac{n-4w}{8}\right| \leq \frac{1}{8}|v-n|+\frac{1}{2}|u-w|, \quad \left|\frac{n-4w}{8}-\frac{m-4l}{8}\right| \leq \frac{1}{8}|m-y|+\frac{1}{2}|w-l|.$

Adding these six inequalities, we get (5.3.27) for $\pi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$, $\wp(\mathfrak{t}_1, \mathfrak{t}_2) = \frac{3}{16}\left(\frac{\mathfrak{t}_1+\mathfrak{t}_2}{2}\right)$. Further, $\varkappa_0 = -1$, $y_0 = 1$ are in $\mathfrak{X}$ such that $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(y_0, \varkappa_0)$. Now, by Corollary 5.3.1, F has a coupled fixed point (0, 0) in $\mathfrak{X}$.

Coupled Common Fixed Points

Now, we establish the existence and uniqueness of coupled common fixed point under the hypotheses of hypotheses of Theorem 5.3.1.

Theorem 5.3.2. In addition to the hypotheses of Theorem 5.3.1, if Assumption 3.2.1 also holds, then, F and g have a unique common coupled fixed point in $\mathfrak{X}$.

Proof. By Theorem 5.3.1, the set of coupled coincidence points of F and g in non-empty. To obtain the result, we first show that if $(\varkappa, y)$, $(\varkappa^*, y^*)$ are coupled coincidence points, then

$$g\varkappa = g\varkappa^* \text{ and } gy = gy^*. \tag{5.3.30}$$

By Assumption 3.2.1, there exists some $(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{X} \times \mathfrak{X}$, so that $(F(\mathfrak{a}, \mathfrak{b}), F(\mathfrak{b}, \mathfrak{a}))$ is comparable to $(F(\varkappa, y), F(y, \varkappa))$ and $(F(\varkappa^*, y^*), F(y^*, \varkappa^*))$. Taking $\mathfrak{a}_0 = \mathfrak{a}$, $\mathfrak{b}_0 = \mathfrak{b}$ and choosing $\mathfrak{a}_1, \mathfrak{b}_1 \in \mathfrak{X}$ so that $g\mathfrak{a}_1 = F(\mathfrak{a}_0, \mathfrak{b}_0)$, $g\mathfrak{b}_1 = F(\mathfrak{b}_0, \mathfrak{a}_0)$. As in Theorem 5.3.1, we can inductively define the sequences $\{g\mathfrak{a}_n\}$ and $\{g\mathfrak{b}_n\}$ so that $g\mathfrak{a}_{n+1} = F(\mathfrak{a}_n, \mathfrak{b}_n)$, $g\mathfrak{b}_{n+1} = F(\mathfrak{b}_n, \mathfrak{a}_n)$. Take $\varkappa_0 = \varkappa$, $y_0 = y$, $\varkappa_0^* = \varkappa$, $y_0^* = y$ and on the same way, define sequences $\{g\varkappa_n\}$, $\{gy_n\}$, $\{g\varkappa_n^*\}$ and $\{gy_n^*\}$. Then, we can easily obtain that

$$g\varkappa_{n+1} = F(\varkappa_n, y_n),\ gy_{n+1} = F(y_n, \varkappa_n),$$

and

$$g\varkappa_{n+1}^* = F(\varkappa_n^*, y_n^*),\ gy_{n+1}^* = F(y_n^*, \varkappa_n^*) \text{ for all } n \geq 0.$$

Since $\big(F(\varkappa, y), F(y, \varkappa)\big) = (g\varkappa_1, gy_1) = (g\varkappa, gy)$ and $\big(F(\mathfrak{a}, \mathfrak{b}), F(\mathfrak{b}, \mathfrak{a})\big) = (g\mathfrak{a}_1, g\mathfrak{b}_1)$ are comparable, then $g\varkappa \preccurlyeq g\mathfrak{a}_1$ and $gy \succcurlyeq g\mathfrak{b}_1$. It is easy to see $(g\varkappa, gy)$ and $(g\mathfrak{a}_n, g\mathfrak{b}_n)$ are comparable, so that $g\varkappa \preccurlyeq g\mathfrak{a}_n$ and $gy \succcurlyeq g\mathfrak{b}_n$ for all $n \geq 1$. Using (5.3.1), we have

$$\pi\left(\frac{G(g\mathfrak{a}_{n+1}, g\varkappa, g\varkappa) + G(g\mathfrak{b}_{n+1}, gy, gy)}{2}\right)$$

$$= \pi\left(\frac{G\big(F(\mathfrak{a}_n, \mathfrak{b}_n), F(\varkappa, y), F(\varkappa, y)\big) + G\big(F(\mathfrak{b}_n, \mathfrak{a}_n), F(y, \varkappa), F(y, \varkappa)\big)}{2}\right)$$

$$\leq \pi\left(\frac{G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy)}{2}\right) - \wp\big(G(g\mathfrak{a}_n, g\varkappa, g\varkappa), G(g\mathfrak{b}_n, gy, gy)\big) \tag{5.3.31}$$

$$\leq \pi\left(\frac{G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy)}{2}\right). \tag{5.3.32}$$

Then, using the monotone property of π, we get

$$G(g\mathfrak{a}_{n+1}, g\varkappa, g\varkappa) + G(g\mathfrak{b}_{n+1}, gy, gy) \leq G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy).$$

Let $\tau_n = G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy)$, then $\{\tau_n\}$ is a monotonic decreasing sequence, thus, there exists some $\tau \geq 0$, such that

$$\lim_{n\to\infty}\tau_n = \lim_{n\to\infty}[G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy)] = \tau.$$

We claim that $\tau = 0$. On the contrary, suppose that $\tau > 0$. Then, $\{G(g\mathfrak{a}_n, g\varkappa, g\varkappa)\}$, $\{G(g\mathfrak{b}_n, gy, gy)\}$ have convergent sub-sequences converging to τ_1, τ_2 (say) respectively.

Considering limit up to sub-sequences as $n \to \infty$ in (5.3.31) and by continuity of π, we get

$$\pi\left(\frac{\tau}{2}\right) \leq \pi\left(\frac{\tau}{2}\right) - \lim_{n\to\infty}\wp\big(G(g\mathfrak{a}_n, g\varkappa, g\varkappa), G(g\mathfrak{b}_n, gy, gy)\big) < \pi\left(\frac{\tau}{2}\right),$$

a contradiction. Therefore, $\tau = 0$, so that

$$\lim_{n\to\infty}[G(g\mathfrak{a}_n, g\varkappa, g\varkappa) + G(g\mathfrak{b}_n, gy, gy)] = 0.$$

Hence, we get $g\mathfrak{a}_n \to g\varkappa$ and $g\mathfrak{b}_n \to gy$ as $n \to \infty$. Similarly, we can get $g\mathfrak{a}_n \to g\varkappa^*$ and $g\mathfrak{b}_n \to gy^*$ as $n \to \infty$. Now, by uniqueness of limit, we have $g\varkappa = g\varkappa^*$ and $gy = gy^*$. Therefore, we proved (5.3.30).

Since $g\varkappa = \text{Ꞙ}(\varkappa, y)$, $gy = \text{Ꞙ}(y, \varkappa)$ and Ꞙ, g commutes, we obtain

$$gg\varkappa = g\text{Ꞙ}(\varkappa, y) = \text{Ꞙ}(g\varkappa, gy) \text{ and } ggy = g\text{Ꞙ}(y, \varkappa) = \text{Ꞙ}(gy, g\varkappa). \quad (5.3.33)$$

Denote $g\varkappa = c$ and $gy = d$, so by (5.3.33), we get

$$gc = \text{Ꞙ}(c, d) \text{ and } gd = \text{Ꞙ}(d, c). \quad (5.3.34)$$

Therefore, (c, d) is a coupled coincidence point of Ꞙ and g.

By (5.3.30) with $r^* = c$ and $y^* = d$, it follows that $gc = gr$ and $gd = gy$, so that

$$gc = c,\ gd = d. \quad (5.3.35)$$

Now, from (5.3.34) and (5.3.35), we have

$$c = gc = \text{Ꞙ}(c, d) \text{ and } d = gd = \text{Ꞙ}(d, c).$$

Therefore, (c, d) is a coupled common fixed point of Ꞙ and g.

For uniqueness, let (e, f) be any coupled common fixed point of Ꞙ and g. Then, by (5.3.30), we get $e = ge = gc = c$ and $f = gf = gd = d$. This completes the proof of our result.

5.4 APPLICATION TO INTEGRAL EQUATIONS

As application of the results produced in section 5.2, we now discuss the existence of solutions of the following system of integral equations:

$$\begin{aligned} \varkappa(t) &= p(t) + \int_0^T L(t,s)\big[f\big(s,\varkappa(s)\big) + k\big(s,y(s)\big)\big]\,ds, \\ y(t) &= p(t) + \int_0^T L(t,s)\big[f\big(s,y(s)\big) + k\big(s,\varkappa(s)\big)\big]\,ds. \end{aligned} \tag{5.4.1}$$

Let Θ_4 be the class of functions $\theta: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions of Luong and Thuan [67]:

(I) θ is increasing;

(II) there exists $\psi \in \Psi$ such that $\theta(r) = \frac{r}{2} - \psi\left(\frac{r}{2}\right)$, for all $r \in \mathbb{R}^+$.

We analyze the system (5.4.1) under the following assumptions:

(i) $f, k: [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous;

(ii) $p: [0, T] \to \mathbb{R}$ is continuous;

(iii) $L: [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is continuous;

(iv) there exist $\lambda > 0$ and $\theta \in \Theta_4$ such that for all $\varkappa, y \in \mathbb{R}$, $y \geq \varkappa$,

$$0 \leq f(s, y) - f(s, \varkappa) \leq \lambda\, \theta(y - \varkappa), \quad 0 \leq k(s, \varkappa) - k(s, y) \leq \lambda\, \theta(y - \varkappa);$$

(v) suppose that $3\lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds \leq \frac{1}{2}$;

(vi) there exist continuous functions $\alpha, \beta: [0, T] \to \mathbb{R}$ such that

$$\alpha(t) \leq p(t) + \int_0^T L(t,s)\left(f\big(s,\alpha(s)\big) + k\big(s,\beta(s)\big)\right) ds,$$

$$\beta(t) \geq p(t) + \int_0^T L(t,s)\left(f\big(s,\beta(s)\big) + k\big(s,\alpha(s)\big)\right) ds.$$

Let $X = C([0, T], \mathbb{R})$ be the set of all continuous functions defined on $[0, T]$ endowed with the following G-metric (which in fact, is G-complete):

$$G(u, v, w) = \sup_{t \in [0,T]}|u(t) - v(t)| + \sup_{t \in [0,T]}|v(t) - w(t)| + \sup_{t \in [0,T]}|w(t) - u(t)| \text{ for all } u, v, w \in X.$$

Endow the set X with the partial order $\preccurlyeq$ defined by:

$\varkappa, y \in X$, $\varkappa \preccurlyeq y \Leftrightarrow \varkappa(t) \leq y(t)$ for all $t \in [0, T]$.

Then, X assumes Assumption 2.1.7 w.r.t. convergence and ordering in $(X, G, \preccurlyeq)$.

Theorem 5.4.1. Under assumptions (i) – (vi), the system (5.4.1) has a solution in $X^2 = \big(C([0, T], \mathbb{R})\big)^2$.

Proof. Define $F: X \times X \to X$ by

$F(\varkappa, y)(t) = p(t) + \int_0^T L(t,s)\big[f\big(s,\varkappa(s)\big) + k\big(s,y(s)\big)\big]ds$, for $t \in [0, T]$ and for all $\varkappa, y \in X$.

We first show that F has MMP.

In fact, for $\varkappa_1 \preccurlyeq \varkappa_2$ and $t \in [0, T]$, we have

$$Ƒ(\varkappa_2, y)(ƫ) - Ƒ(\varkappa_1, y)(ƫ) = \int_0^T L(ƫ, s)\big[f\big(s, \varkappa_2(s)\big) - f\big(s, \varkappa_1(s)\big)\big]ds.$$

Now, for $\varkappa_1(ƫ) \leq \varkappa_2(ƫ)$ for all $ƫ \in [0, T]$, then by assumption (iv), $f(s, \varkappa_2(s)) \succcurlyeq f(s, \varkappa_1(s))$. Then, $Ƒ(\varkappa_2, y)(ƫ) \geq Ƒ(\varkappa_1, y)(ƫ)$ for all $ƫ \in [0, T]$, so that $Ƒ(\varkappa_1, y) \preccurlyeq Ƒ(\varkappa_2, y)$.

Similarly, for $y_1 \preccurlyeq y_2$ and $ƫ \in [0, T]$, we obtain

$$Ƒ(\varkappa, y_1)(ƫ) - Ƒ(\varkappa, y_2)(ƫ) = \int_0^T L(ƫ, s)\big[ƙ\big(s, y_1(s)\big) - ƙ\big(s, y_2(s)\big)\big]ds.$$

Having $y_1(ƫ) \leq y_2(ƫ)$, so by (iv), $ƙ\big(s, y_1(s)\big) \succcurlyeq ƙ\big(s, y_2(s)\big)$.

Then $Ƒ(\varkappa, y_1)(ƫ) \geq Ƒ(\varkappa, y_2)(ƫ)$ for all $ƫ \in [0, T]$, so that $Ƒ(\varkappa, y_1) \succcurlyeq Ƒ(\varkappa, y_2)$. Hence, Ƒ has MMP.

Now, for $\varkappa, y, z, u, v, w \in \mathfrak{X}$ with $\varkappa \succcurlyeq u \succcurlyeq w$, $y \preccurlyeq v \preccurlyeq z$, we estimate the quantity $G(Ƒ(\varkappa, y), Ƒ(u, v), Ƒ(w, z))$. (Note that, here $\varkappa, y, z, u, v, w \in \mathfrak{X}$ are functions of $ƫ \in [0, T]$).

By MMP of Ƒ, we get $Ƒ(w, z) \preccurlyeq Ƒ(u, v) \preccurlyeq Ƒ(\varkappa, y)$, then, we obtain that

$$\begin{aligned}
&G(Ƒ(\varkappa, y), Ƒ(u, v), Ƒ(w, z)) \\
&\quad = \sup_{ƫ \in [0,T]}|Ƒ(\varkappa, y)(ƫ) - Ƒ(u,v)(ƫ)| + \sup_{ƫ \in [0,T]}|Ƒ(u,v)(ƫ) - Ƒ(w, z)(ƫ)| \\
&\qquad + \sup_{ƫ \in [0,T]}|Ƒ(w, z)(ƫ) - Ƒ(\varkappa, y)(ƫ)| \\
&\quad = \sup_{ƫ \in [0,T]}\big(Ƒ(\varkappa, y)(ƫ) - Ƒ(u,v)(ƫ)\big) + \sup_{ƫ \in [0,T]}\big(Ƒ(u,v)(ƫ) - Ƒ(w, z)(ƫ)\big) \\
&\qquad + \sup_{ƫ \in [0,T]}\big(Ƒ(\varkappa, y)(ƫ) - Ƒ(w, z)(ƫ)\big).
\end{aligned}$$

Also, for all $ƫ \in [0, T]$, by assumption (iv), we get

$$\begin{aligned}
Ƒ(\varkappa, y) - Ƒ(u, v) &= \int_0^T L(ƫ, s)\big[f\big(s, \varkappa(s)\big) - f\big(s, u(s)\big)\big]ds \\
&\quad + \int_0^T L(ƫ, s)\big[ƙ\big(s, y(s)\big) - ƙ\big(s, v(s)\big)\big]ds \\
&\leq \lambda \int_0^T L(ƫ, s)\Big[\theta\Big(\varkappa(s) - u(s)\Big) + \theta(v(s) - y(s))\Big]ds. \qquad (5.4.2)
\end{aligned}$$

As θ is an increasing function and $\varkappa \succcurlyeq u \succcurlyeq w$, $y \preccurlyeq v \preccurlyeq z$, we have

$\theta(\varkappa(s) - u(s)) \leq \theta(\sup_{ƫ \in I}|\varkappa(ƫ) - u(ƫ)|)$, $\theta(v(s) - y(s)) \leq \theta(\sup_{t \in I}|v(ƫ) - y(ƫ)|)$, so that, using (5.4.2), we get

$$\begin{aligned}
&\big|Ƒ(\varkappa, y) - Ƒ(u,v)\big| \\
&\quad \leq \lambda \int_0^T L(ƫ, s)[\theta(\sup_{ƫ \in I}|\varkappa(ƫ) - u(ƫ)|) + \theta(\sup_{ƫ \in I}|v(ƫ) - y(ƫ)|)]ds. \qquad (5.4.3)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\big|Ƒ(\varkappa, y) - Ƒ(w, z)\big| \\
&\quad \leq \lambda \int_0^T L(ƫ, s)[\theta(\sup_{ƫ \in I}|\varkappa(ƫ) - w(ƫ)|) + \theta(\sup_{ƫ \in I}|z(ƫ) - y(ƫ)|)]ds, \qquad (5.4.4)
\end{aligned}$$

$$\big|Ƒ(w, z) - Ƒ(u, v)\big|$$

$$\leq \lambda \int_0^T L(t,s)[\theta(\sup_{t\in I}|u(t) - w(t)|) + \theta(\sup_{t\in I}|z(t) - v(t)|)]ds. \quad (5.4.5)$$

Adding (5.4.3), (5.4.4) and (5.4.5) and taking supremum w.r.t. t, we get

G(F(ϰ, y), F(u, v), F(w, z))

$$\leq \lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds \cdot \begin{bmatrix} \theta(\sup_{t\in I}|\varkappa(t) - u(t)|) + \theta(\sup_{t\in I}|\varkappa(t) - w(t)|) \\ +\theta(\sup_{t\in I}|u(t) - w(t)|) \end{bmatrix}$$

$$+ \lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds \cdot \begin{bmatrix} \theta(\sup_{t\in I}|v(t) - y(t)|) + \theta(\sup_{t\in I}|z(t) - y(t)|) \\ +\theta(\sup_{t\in I}|z(t) - v(t)|) \end{bmatrix}. \quad (5.4.6)$$

Also, since θ is increasing, we have

$$\theta(\sup_{t\in I}|\varkappa(t) - u(t)|) \leq \theta(G(\varkappa,u,w)),\ \theta(\sup_{t\in I}|\varkappa(t) - w(t)|) \leq \theta(G(\varkappa,u,w)),$$
$$\theta(\sup_{t\in I}|u(t) - w(t)|) \leq \theta(G(\varkappa,u,w)).$$

Similarly,

$$\theta(\sup_{t\in I}|v(t) - y(t)|) \leq \theta(G(y,v,z)),\quad \theta(\sup_{t\in I}|z(t) - y(t)|) \leq \theta(G(y,v,z)),$$
$$\theta(\sup_{t\in I}|z(t) - v(t)|) \leq \theta(G(y,v,z)).$$

Then, by (5.4.6) and using assumption (v), we get

G(F(ϰ, y), F(u, v), F(w, z))

$$\leq \lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds\ .\ 3\,\theta(G(\varkappa,u,w)) + \lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds\ .\ 3\,\theta(G(y,v,z))$$

$$= 3\lambda \sup_{t\in[0,T]} \int_0^T L(t,s)ds\ .\ (\theta(G(\varkappa,u,w)) + \theta(G(y,v,z)))$$

$$\leq \frac{\theta(G(\varkappa,u,w)) + \theta(G(y,v,z))}{2}. \quad (5.4.7)$$

As θ is increasing, we have

$$\theta(G(\varkappa,u,w)) \leq \theta(G(\varkappa,u,w) + G(y,v,z)),\ \theta(G(y,v,z)) \leq \theta(G(\varkappa,u,w) + G(y,v,z))$$

and so $\frac{\theta(G(\varkappa,u,w))+\theta(G(y,v,z))}{2} \leq \theta(G(\varkappa,u,w) + G(y,v,z))$

$$= \frac{G(\varkappa,u,w)+G(y,v,z)}{2} - \psi\left(\frac{G(\varkappa,u,w)+G(y,v,z)}{2}\right), \quad (5.4.8)$$

by definition of θ. Now, using (5.4.7) and (5.4.8), we get

$$G(F(\varkappa, y), F(u, v), F(w, z)) \leq \frac{G(\varkappa,u,w)+G(y,v,z)}{2} - \psi\left(\frac{G(\varkappa,u,w)+G(y,v,z)}{2}\right),$$

which is the actually the condition (5.2.28) of Corollary 5.2.4.

Let α, β be the functions in assumption (vi), therefore, we obtain that $\alpha \preccurlyeq F(\alpha, \beta)$ and $\beta \succcurlyeq F(\beta, \alpha)$. Now, by Corollary 5.2.4, there exist ϰ, y ∈ X such that ϰ = F(ϰ, y), y = F(y, ϰ), so that (ϰ, y) is a solution of the system (5.4.1).

FRAMEWORK OF CHAPTER - VI

In this chapter, we give a new technique to compute coupled coincidence points in various spaces. Also, we rectify certain errors present in some recent papers.

PUBLISHED WORK:

(1) Chinese Journal of Mathematics, vol. 2014, Article ID 652107, 6 pages, 2014.

(2) Communications in Nonlinear Analysis, 2(1) (2016), 84–85.

(3) Bulletin of the Iranian Mathematical Society, 42(1) (2016), pp. 49–52.

CONFERENCES:

(1) One paper presented in the National Conference CPMSED-2015 held on Nov 28, 2015 at JNU, New Delhi organized by Krishi Sanskriti and published in the conference proceedings.

(2) One paper presented in the National Conference RSTTMI 2016 held during March 05-07, 2016 at YMCAUST, Faridabad and published in the conference proceedings.

(3) One paper presented in the International Conference RAMSA-2016 held during Dec 08-10, 2016 at JIIT, Noida.

(4) One paper presented in the International Conference ICSCMM-17 held during Dec 22-23, 2017 at KIET Group of Institutions, Ghaziabad (U.P.) and published in the conference proceedings:

Malaya Journal of Matematik, S(1) (2018), pp. 5-13.

(Some part of this paper has already utilized in Chapter – V and the remaining part is used in this chapter).

CHAPTER – VI

A NEW TECHNIQUE AND ERRORS IN SOME RECENT PAPERS

In this chapter, we study a new technique to compute coupled coincidence points in various spaces. Also, we rectify some errors present in the recent papers on coupled coincidence and coupled common fixed points in some spaces. This chapter consists of eight sections. Section 6.1 gives a brief introduction to some previous results. In section 6.2, we discuss a new technique to compute coupled coincidence points. In section 6.3, using the technique given in section 6.2 we improve some recent coupled coincidence point results in POMS. Section 6.4 consists of the generalization of a recent coupled coincidence point result in POMPMS by using the technique given in section 6.2. In section 6.5, using the technique given in section 6.2, we generalize Theorem 5.3.1. Section 6.6 consists of some remarks on some recent papers concerning coupled coincidence points. In section 6.7, we point out and rectify an error in a recent paper on probabilistic φ – contraction in PGM-spaces. In section 6.8, we point out and rectify some errors in a recent paper on weakly related mappings in POMS.

Author's Original Contributions In This Chapter Are:

Theorems: 6.2.1, 6.2.2, 6.3.3, 6.4.2, 6.5.1, 6.6.2, 6.6.4, 6.6.6, 6.6.8.

Examples: 6.2.1, 6.7.2, 6.7.3, 6.8.2, 6.8.4.

Remarks: 6.2.1, 6.3.1, 6.4.1, 6.5.1, 6.6.1, 6.6.2, 6.6.3, 6.6.4, 6.6.5, 6.7.1, 6.8.1, 6.8.2, 6.8.3.

6.1 INTRODUCTION

Recently, Haghi et al. [165] showed that certain common fixed point results are not true generalizations of the fixed point results. For proving this, Haghi et al. [165] proved and utilized the following lemma:

Lemma 6.1.1. ([165]). Let $\mathfrak{X}$ be a non-empty set and g: $\mathfrak{X} \to \mathfrak{X}$ be a mapping, then there exists a subset A of $\mathfrak{X}$ such that g(A) = g($\mathfrak{X}$) and the mapping g: A $\to \mathfrak{X}$ is one-to-one.

The technique used by Haghi et al. [165] was extended by Sintunavarat et al. [166] to obtain coupled coincidence points in intuitionistic fuzzy normed spaces.

Hussain et al. [167] used the technique of Sintunavarat et al. [166] to generalize results due to Lakshmikantham and Ćirić [59], Choudhury and Kundu [60] and Alotaibi and Alsulami [68]. As a matter of fact, Hussain et al. [167] in their results assumed the completeness of the range subspace of the involved self mapping and relaxed the assumptions of compatibility (and, hence of commutativity) and the completeness of the space $\mathfrak{X}$.

For the sake of convenience, we recall some notions stated already in the previous chapters.

Assumption 2.1.7 ([55]). $\mathfrak{X}$ has the property:

(i) "if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset \mathfrak{X}$ converges to $\varkappa$, then $\varkappa_n \preccurlyeq \varkappa$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset \mathfrak{X}$ converges to y, then $y \preccurlyeq y_n$ for all n".

Assumption 2.1.8 ([56]). $\mathfrak{X}$ has the property:

(i) "if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset \mathfrak{X}$ converges to $\varkappa$, then $g\varkappa_n \preccurlyeq g\varkappa$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset \mathfrak{X}$ converges to y, then $gy \preccurlyeq gy_n$ for all n".

Property (P2): "There exist two elements $\varkappa_0, y_0 \in \mathfrak{X}$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$".

The following result of Hussain et al. [167] generalize Theorems 2.1.16 and 2.1.17:

Theorem 6.1.1 ([167]). Let $(\mathfrak{X}, \preccurlyeq, d)$ be a POMS. Assume there is a function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(f) < f$ for all $f > 0$ and $\lim_{r \to f^+} \varphi(r) < f$ for each $f > 0$. Also, suppose that $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings such that F has MgMP on $\mathfrak{X}$, $g(\mathfrak{X})$ is complete and F, g satisfies (2.1.16), that is

$$d\big(F(\varkappa, y),\ F(u, v)\big) \le \varphi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right), \tag{6.1.1}$$

for all $\varkappa$, y, u, v $\in \mathfrak{X}$ for which $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$. Further suppose that $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$, g is continuous and either

(a) F is continuous, or (b) $\mathfrak{X}$ assumes Assumption 2.1.7.

If $\mathfrak{X}$ has the property (P2), then F and g have a coupled coincidence point in $\mathfrak{X}$.

As in Definition 2.1.13, denote by Φ_1, the class of functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy:

(φ_1) φ is continuous and non-decreasing;

(φ_2) φ(f) = 0 if and only if f = 0;

(φ_3) φ(f + s) ≤ φ(f) + φ(s), for all f, s ∈ $\mathbb{R}^+$.

Again as in Definition 2.1.14, let Ψ denote the class of functions ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ which satisfy: (i_ψ) "$\lim_{f \to r} \psi(f) > 0$ for all r > 0 and $\lim_{f \to 0+} \psi(f) = 0$".

Hussain et al. [167] also generalized Theorem 2.1.20 under the following result:

Theorem 6.1.2 ([167]). Let (X, ≼, d) be a POMS and F: X × X → X, g: X → X be two mapping such that F has MgMP on X. Suppose there exist $\varphi \in \Phi_1$, $\psi \in \Psi$ such that for all ϰ, y, u, v ∈ X with gϰ ≽ gu and gy ≼ gv, the mappings F, g satisfies (2.1.19), that is

$$\varphi(d(F(\varkappa, y), F(u, v))) \leq \frac{1}{2}\varphi(d(g\varkappa, gu) + d(gy, gv)) - \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right). \quad (6.1.2)$$

Assume that F(X × X) ⊆ g(X), g is continuous and g(X) is complete. Also, suppose either

(a) F is continuous, or (b) X assumes Assumption 2.1.7.

If X has the property (P2), then F and g have a coupled coincidence point in X.

6.2 A NEW TECHNIQUE TO COMPUTE COUPLED COINCIDENCE POINTS

In this section, we develop a technique that generalizes and improves the technique introduced by Sintunavarat et al. [166] which was used by Hussain et al [167].

Now, we give our first main result as follows:

Theorem 6.2.1. Let (X, ≼, d) be a POMS and F: X × X → X, g: X → X be two mappings. Suppose there exists a function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that φ(f) < f for all f > 0 and $\lim_{r \to f^+} \varphi(r) < f$ for each f > 0. Also, assume that F(X × X) ⊆ g(X), g(X) is complete, F has MgMP on X and F, g satisfy (6.1.1), that is

$$d\big(F(\varkappa, y), F(u, v)\big) \leq \varphi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right), \quad (6.2.1)$$

for all ϰ, y, u, v ∈ X for which gϰ ≼ gu and gy ≽ gv. Also, suppose either

(a) F and g both are continuous, or (b) X assumes Assumption 2.1.7.

If X has the property (P2), then F and g have a coupled coincidence point in X.

Proof. As $\mathfrak{X}$ has the property (P2), there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. As $F(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$ and F has MgMP in $\mathfrak{X}$, then as in the proof of Theorem 3.2.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in $\mathfrak{X}$ with

$$g\varkappa_{n+1} = F(\varkappa_n, y_n), \quad gy_{n+1} = F(y_n, \varkappa_n) \text{ for all } n \geq 0, \tag{6.2.2}$$

and

$$g\varkappa_n \preccurlyeq g\varkappa_{n+1}, \quad gy_n \succcurlyeq gy_{n+1} \text{ for all } n \geq 0. \tag{6.2.3}$$

Suppose either $g\varkappa_{n+1} = F(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = F(y_n, \varkappa_n) \neq gy_n$, otherwise we obtain directly the coupled coincidence point of F and g.

Let $R_n = d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})$. We now show that

$$R_n \leq 2\varphi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.4}$$

As $g\varkappa_n \preccurlyeq g\varkappa_{n+1}$ and $gy_n \succcurlyeq gy_{n+1}$ for all $n \geq 1$, by (6.2.1) and (6.2.2), we get

$$d(g\varkappa_n, g\varkappa_{n+1}) = d(F(\varkappa_{n-1}, y_{n-1}), F(\varkappa_n, y_n))$$

$$\leq \varphi\left(\frac{d(g\varkappa_{n-1}, g\varkappa_n) + d(gy_{n-1}, gy_n)}{2}\right) = \varphi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.5}$$

Similarly, for all $n \geq 1$, we obtain that

$$d(g\varkappa_n, g\varkappa_{n+1}) \leq \varphi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.6}$$

Combining (6.2.5) and (6.2.6), we get (6.2.4). Also, as $\varphi(t) < t$ for $t > 0$, by (6.2.4), it follows that $\{R_n\}$ is a monotone decreasing sequence of non-negative terms. So, there exists some $R \geq 0$, such that $\lim_{n\to\infty} R_n = R$. We claim $R = 0$. Suppose, on the contrary that $R > 0$. Letting $n \to \infty$ in (6.2.4) and using $\lim_{r \to t^+} \varphi(r) < t$ for all $t > 0$, we can get $R = \lim_{n\to\infty} R_n \leq 2 \lim_{n\to\infty} \varphi\left(\frac{R_{n-1}}{2}\right) < 2\,\frac{R}{2} = R$, a contradiction. Therefore, $R = 0$, so that we have

$$\lim_{n\to\infty} [d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})] = \lim_{n\to\infty} R_n = 0. \tag{6.2.7}$$

Next, we prove that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let at least one of $\{g\varkappa_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every $k \in \mathbb{N}$,

$$m(k) > l(k) \geq k$$

and

$$d_k = d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)}) \geq \varepsilon. \tag{6.2.8}$$

Now corresponding to $l(k)$ there exists a smallest $m(k) \in \mathbb{N}$ for which (6.2.8) holds. Then,

$$d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(gy_{l(k)}, gy_{m(k)-1}) < \varepsilon. \tag{6.2.9}$$

Also, using (6.2.8), (6.2.9) and the triangle inequality, for all $k > 0$, we have

$$\varepsilon \leq d_k \leq d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(g\varkappa_{m(k)-1}, g\varkappa_{m(k)})$$

$$+ d(gy_{l(k)}, gy_{m(k)-1}) + d(gy_{m(k)-1}, gy_{m(k)})$$

$$= d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(gy_{l(k)}, gy_{m(k)-1}) + R_{m(k)-1}$$
$$< \varepsilon + R_{m(k)-1}.$$

Letting $k \to \infty$ in the above inequality and using (6.2.7), we have

$$\lim_{k\to\infty} d_k = \varepsilon. \quad (6.2.10)$$

Again by triangle inequality, for all $k > 0$, we have

$$d_k = d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)})$$
$$\leq d(g\varkappa_{l(k)}, g\varkappa_{l(k)+1}) + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(g\varkappa_{m(k)+1}, g\varkappa_{m(k)})$$
$$+ d(gy_{l(k)}, gy_{l(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)})$$
$$= d(g\varkappa_{l(k)}, g\varkappa_{l(k)+1}) + d(gy_{l(k)}, gy_{l(k)+1}) + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1})$$
$$+ d(gy_{l(k)+1}, gy_{m(k)+1}) + d(g\varkappa_{m(k)+1}, g\varkappa_{m(k)}) + d(gy_{m(k)+1}, gy_{m(k)}).$$

Hence, for all $k > 0$, we have

$$d_k \leq R_{l(k)} + R_{m(k)} + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1}). \quad (6.2.11)$$

Using (6.2.1), (6.2.2), (6.2.3) and (6.2.8), for all $k > 0$, we have

$$d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) = d(F(\varkappa_{l(k)}, y_{l(k)}), F(\varkappa_{m(k)}, y_{m(k)}))$$
$$\leq \varphi\left(\frac{d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)})}{2}\right) = \varphi\left(\frac{d_k}{2}\right). \quad (6.2.12)$$

Similarly,

$$d(gy_{l(k)+1}, gy_{m(k)+1}) = d(F(y_{l(k)}, \varkappa_{l(k)}), F(y_{m(k)}, \varkappa_{m(k)}))$$
$$\leq \varphi\left(\frac{d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)})}{2}\right) = \varphi\left(\frac{d_k}{2}\right). \quad (6.2.13)$$

Putting (6.2.12) and (6.2.13) in (6.2.11), for all $k > 0$, we obtain that

$$d_k \leq R_{l(k)} + R_{m(k)} + 2\varphi\left(\frac{d_k}{2}\right). \quad (6.2.14)$$

Letting $k \to \infty$ in (6.2.14) and using (6.2.7), (6.2.8) and (6.2.10), we obtain that

$$\varepsilon \leq 2 \lim_{k\to\infty} \varphi\left(\frac{d_k}{2}\right) < 2\,\frac{\varepsilon}{2} = \varepsilon, \quad (6.2.15)$$

a contradiction. Therefore, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences.

By the completeness of $g(X)$, there exist $\varkappa$, y in X such that

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \quad \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = g(y). \quad (6.2.16)$$

We finally show that $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$.

Suppose that assumption (a) holds.

By Lemma 6.1.1, there exists some $A \subseteq X$ such that $g(A) = g(X)$ and the mapping $g: A \to X$ is one-to-one. Define a mapping $H: g(A) \times g(A) \to X$ by

$$H(ga, gb) = F(a, b) \text{ for } ga, gb \in g(A). \quad (6.2.17)$$

Since g is one-to-one on A, so Ή is well defined.

Using (6.2.16) and (6.2.17), we get

$$\lim_{n\to\infty} Ή(g\varkappa_n, gy_n) = \lim_{n\to\infty} Ƒ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \tag{6.2.18}$$

and $$\lim_{n\to\infty} Ή(gy_n, g\varkappa_n) = \lim_{n\to\infty} Ƒ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \tag{6.2.19}$$

As Ƒ and g are continuous, Ή is also continuous. Then, by (6.2.18) and (6.2.19), we get

$$Ή(g\varkappa, gy) = g\varkappa \text{ and } Ή(gy, g\varkappa) = gy. \tag{6.2.20}$$

Using (6.2.17) and (6.2.20), we get $Ƒ(\varkappa, y) = g\varkappa$ and $Ƒ(y, \varkappa) = gy$.

Now, suppose that assumption (b) holds. By (6.2.3) and (6.2.16), we have $\{g\varkappa_n\}$ is a non-decreasing sequence converging to $g\varkappa$ and $\{gy_n\}$ is a non-increasing sequence converging to gy. Hence, by assumption, for all $n \geq 0$, we have

$$g\varkappa_n \preccurlyeq g\varkappa \text{ and } gy \preccurlyeq gy_n. \tag{6.2.21}$$

We suppose that $(g\varkappa_n, gy_n) \neq (g\varkappa, gy)$ for all $n \geq 0$, otherwise, we can directly obtain a coupled coincidence point of Ƒ and g.

Now, using (6.2.1) for (6.2.21), we get

$$ḑ\big(Ƒ(\varkappa_n, y_n), Ƒ(\varkappa, y)\big) \leq \varphi\left(\frac{ḑ(g\varkappa_n, g\varkappa) + ḑ(gy_n, gy)}{2}\right). \tag{6.2.22}$$

By triangle inequality, we have

$$ḑ\big(g\varkappa, Ƒ(\varkappa, y)\big) \leq ḑ\big(g\varkappa, Ƒ(\varkappa_n, y_n)\big) + ḑ(Ƒ(\varkappa_n, y_n), Ƒ(\varkappa, y)). \tag{6.2.23}$$

Inserting (6.2.22) in (6.2.23) and letting $n \to \infty$, we get

$$ḑ\big(g\varkappa, Ƒ(\varkappa, y)\big) \leq \lim_{n\to\infty}\left\{ḑ\big(g\varkappa, Ƒ(\varkappa_n, y_n)\big) + \varphi\left(\frac{ḑ(g\varkappa_n, g\varkappa) + ḑ(gy_n, gy)}{2}\right)\right\}.$$

Using (6.2.16) and the property of φ, we obtain that $ḑ(g\varkappa, Ƒ(\varkappa, y)) \leq 0$, so that $g\varkappa = Ƒ(\varkappa, y)$. Similarly, we can get $gy = Ƒ(y, \varkappa)$. Hence, in both the cases Ƒ and g have a coupled coincidence point in Ҳ.

Now, we improve Theorem 6.1.2 as follows:

Theorem 6.2.2. Let (Ҳ, ≼, ḑ) be a POMS and Ƒ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be two mappings such that Ƒ has MgMP on Ҳ. Also, assume there exist $\phi \in \Phi_1$ and $\psi \in \Psi$ such that (6.1.2) holds, that is

$$\phi\left(ḑ\big(Ƒ(\varkappa, y), Ƒ(u, v)\big)\right) \leq \frac{1}{2}\phi\big(ḑ(g\varkappa, gu) + ḑ(gy, gv)\big) - \psi\left(\frac{ḑ(g\varkappa, gu) + ḑ(gy, gv)}{2}\right), \tag{6.2.24}$$

for all $\varkappa$, y, u, v ∈ Ҳ for which $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$. Also, suppose g(Ҳ) is complete in Ҳ, Ƒ(Ҳ × Ҳ) ⊆ g(Ҳ) and either

(a) Ғ and g both are continuous or (b) Ӿ assumes Assumption 2.1.7.

If Ӿ has property (P2), then, Ғ and g have a coupled coincidence point in Ӿ.

Proof. As Ӿ has the property (P2), there exist $\varkappa_0$, y_0 in Ӿ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Also, as $F(X \times X) \subseteq g(X)$ and Ғ has MgMP on Ӿ, then as in the proof of Theorem 3.2.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in Ӿ such that

$$g\varkappa_{n+1} = F(\varkappa_n, y_n),\ gy_{n+1} = F(y_n, \varkappa_n) \text{ for all } n \geq 0 \tag{6.2.25}$$

and $$g\varkappa_n \preccurlyeq g\varkappa_{n+1},\ gy_n \succcurlyeq gy_{n+1} \text{ for all } n \geq 0. \tag{6.2.26}$$

We suppose either $g\varkappa_{n+1} = F(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = F(y_n, \varkappa_n) \neq gy_n$, otherwise the result holds trivially.

Let $R_n = d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})$. We now show that

$$\phi(R_n) \leq \phi(R_{n-1}) - 2\psi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.27}$$

Since $g\varkappa_n \preccurlyeq g\varkappa_{n+1}$ and $gy_n \succcurlyeq gy_{n+1}$ for all $n \geq 1$, by (6.2.24) and (6.2.25), we have

$$\phi(d(g\varkappa_n, g\varkappa_{n+1})) = \phi(d(F(\varkappa_{n-1}, y_{n-1}), F(\varkappa_n, y_n)))$$

$$\leq \frac{1}{2}\phi(d(g\varkappa_{n-1}, g\varkappa_n) + d(gy_{n-1}, gy_n)) - \psi\left(\frac{d(g\varkappa_{n-1}, g\varkappa_n)+d(gy_{n-1}, gy_n)}{2}\right)$$

$$= \frac{1}{2}\phi(R_{n-1}) - \psi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.28}$$

Similarly, for $n \geq 1$, we get

$$\phi\big(d(gy_n, gy_{n+1})\big) \leq \frac{1}{2}\phi(R_{n-1}) - \psi\left(\frac{R_{n-1}}{2}\right). \tag{6.2.29}$$

Combining (6.2.28), (6.2.29) and using (φ_3), we obtain (6.2.27), that is,

$$\phi(R_n) \leq \phi(R_{n-1}) - 2\psi\left(\frac{R_{n-1}}{2}\right) \leq \phi(R_{n-1}), \text{ for all } n \geq 1,$$

then, since ϕ is non-decreasing, we obtain that $R_n \leq R_{n-1}$, so that $\{R_n\}$ is a monotone decreasing sequence of non-negative numbers. So, there exists some $R \geq 0$, such that $\lim_{n\to\infty} R_n = R$. We claim that $R = 0$. Suppose on the contrary that $R > 0$. Letting $n \to \infty$ in (6.2.27) and using $\lim_{t \to r}\psi(t) > 0$ for all $r > 0$ with the continuity of ϕ, we obtain that

$$\phi(R) = \lim_{n\to\infty}\phi(R_n) \leq \lim_{n\to\infty}\left[\phi(R_{n-1}) - 2\psi\left(\frac{R_{n-1}}{2}\right)\right]$$

$$= \phi(R) - 2\lim_{R_{n-1}\to R}\psi\left(\frac{R_{n-1}}{2}\right) < \phi(R),$$

a contradiction. Thus, $R = 0$, so that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}[d(g\varkappa_n, g\varkappa_{n+1}) + d(gy_n, gy_{n+1})] = 0. \tag{6.2.30}$$

Next, we claim that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let at least one of sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every $k \in \mathbb{N}$,

$$m(k) > l(k) \geq k$$

and

$$d_k = d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)}) \geq \varepsilon. \tag{6.2.31}$$

Now, corresponding to $l(k)$ there exists a smallest $m(k) \in \mathbb{N}$ for which (6.2.31) holds. Then,

$$d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(gy_{l(k)}, gy_{m(k)-1}) < \varepsilon. \tag{6.2.32}$$

Also, using (6.2.31), (6.2.32) and triangle inequality, for all $k > 0$, we have

$$\begin{aligned}\varepsilon \leq d_k &\leq d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(g\varkappa_{m(k)-1}, g\varkappa_{m(k)}) \\ &\quad + d(gy_{l(k)}, gy_{m(k)-1}) + d(gy_{m(k)-1}, gy_{m(k)}) \\ &= d(g\varkappa_{l(k)}, g\varkappa_{m(k)-1}) + d(gy_{l(k)}, gy_{m(k)-1}) + R_{m(k)-1} \\ &< \varepsilon + R_{m(k)-1},\end{aligned}$$

on letting $k \to \infty$ and using (6.2.30), we obtain

$$\lim_{k\to\infty} d_k = \varepsilon. \tag{6.2.33}$$

Again using triangle inequality, for all $k > 0$, we get

$$\begin{aligned}d_k &= d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)}) \\ &\leq d(g\varkappa_{l(k)}, g\varkappa_{l(k)+1}) + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(g\varkappa_{m(k)+1}, g\varkappa_{m(k)}) \\ &\quad + d(gy_{l(k)}, gy_{l(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}) \\ &= d(g\varkappa_{l(k)}, g\varkappa_{l(k)+1}) + d(gy_{l(k)}, gy_{l(k)+1}) \\ &\quad + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1}) \\ &\quad + d(g\varkappa_{m(k)+1}, g\varkappa_{m(k)}) + d(gy_{m(k)+1}, gy_{m(k)}) \\ &= R_{l(k)} + R_{m(k)} + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1}).\end{aligned}$$

Now, using the properties of ϕ, for all $k > 0$, we have

$$\begin{aligned}\phi(d_k) &\leq \phi\big(R_{l(k)} + R_{m(k)} + d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1}) + d(gy_{l(k)+1}, gy_{m(k)+1})\big) \\ &\leq \phi\big(R_{l(k)} + R_{m(k)}\big) + \phi\big(d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1})\big) + \phi\big(d(gy_{l(k)+1}, gy_{m(k)+1})\big).\end{aligned} \tag{6.2.34}$$

Using (6.2.24), (6.2.25), (6.2.26) and (6.2.31), for all $k > 0$, we have

$$\begin{aligned}\phi\big(d(g\varkappa_{l(k)+1}, g\varkappa_{m(k)+1})\big) &= \phi\left(d\left(F(\varkappa_{l(k)}, y_{l(k)}), F(\varkappa_{m(k)}, y_{m(k)})\right)\right) \\ &\leq \frac{1}{2}\phi\left(d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)})\right) - \psi\left(\frac{d(g\varkappa_{l(k)}, g\varkappa_{m(k)}) + d(gy_{l(k)}, gy_{m(k)})}{2}\right)\end{aligned}$$

$= \frac{1}{2}\, \phi(d_k) - \psi\left(\frac{d_k}{2}\right)$. (6.2.35)

Similarly,

$$\phi\big(d(gy_{l(k)+1}, gy_{m(k)+1})\big) \le \frac{1}{2}\, \phi(d_k) - \psi\left(\frac{d_k}{2}\right). \quad (6.2.36)$$

Inserting (6.2.35) and (6.2.36) in (6.2.34), for all $k > 0$, we obtain that

$$\phi(d_k) \le \phi(R_{l(k)} + R_{m(k)}) + \phi(d_k) - 2\, \psi\left(\frac{d_k}{2}\right). \quad (6.2.37)$$

Letting $k \to \infty$ in (6.2.37) and using (6.2.30), (6.2.31) and (6.2.33), we obtain that

$$\phi(\varepsilon) \le \phi(0) + \phi(\varepsilon) - 2 \lim_{k \to \infty} \psi\left(\frac{d_k}{2}\right) < \phi(\varepsilon),$$

a contradiction. Hence, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. As $g(X)$ is complete, there exist $\varkappa, y \in X$ such that

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa,\ \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \quad (6.2.38)$$

Finally, we show that $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$.

Suppose that assumption (a) holds.

By Lemma 6.1.1, there exists some $A \subseteq X$ such that $g(A) = g(X)$ and the mapping $g: A \to X$ is one-to-one. Define a mapping $H: g(A) \times g(A) \to X$ by

$$H(ga, gb) = F(a, b) \text{ for all } ga, gb \in g(A). \quad (6.2.39)$$

As g is one-to-one on A, so H is well defined.

By (6.2.38) and (6.2.39), we get

$$\lim_{n\to\infty} H(g\varkappa_n, gy_n) = \lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \quad (6.2.40)$$

$$\lim_{n\to\infty} H(gy_n, g\varkappa_n) = \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \quad (6.2.41)$$

Now, the continuity of F and g implies the continuity of H. Then, by (6.2.40) and (6.2.41), we obtain that

$$H(g\varkappa, gy) = g\varkappa \text{ and } H(gy, g\varkappa) = gy. \quad (6.2.42)$$

Using (6.2.39) and (6.2.42), we get $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

Now, suppose that assumption (b) holds. Then using (6.2.26) and (6.2.38), for all $n \ge 0$, we have

$$g\varkappa_n \preccurlyeq g\varkappa \text{ and } gy \preccurlyeq gy_n. \quad (6.2.43)$$

We suppose that $(g\varkappa_n, gy_n) \neq (g\varkappa, gy)$ for all $n \ge 0$, otherwise, the result follows trivially. Now, using (6.2.24) for (6.2.43), we get

$$\phi\left(d\big(F(\varkappa_n, y_n), F(\varkappa, y)\big)\right)$$
$$\le \frac{1}{2}\phi\big(d(g\varkappa_n, g\varkappa) + d(gy_n, gy)\big) - \psi\left(\frac{d(g\varkappa_n, g\varkappa) + d(gy_n, gy)}{2}\right). \quad (6.2.44)$$

By triangle inequality, we have

$$ d(g\varkappa, F(\varkappa, y)) \leq d(g\varkappa, F(\varkappa_n, y_n)) + d(F(\varkappa_n, y_n), F(\varkappa, y)). $$

Using the properties of ϕ, we get

$$ \phi\left(d(g\varkappa, F(\varkappa, y))\right) \leq \phi\left(d(g\varkappa, F(\varkappa_n, y_n)) + d(F(\varkappa_n, y_n), F(\varkappa, y))\right) $$

$$ \leq \phi\left(d(g\varkappa, F(\varkappa_n, y_n))\right) + \phi\left(d(F(\varkappa_n, y_n), F(\varkappa, y))\right). \qquad (6.2.45) $$

Inserting (6.2.44) in (6.2.45), we get

$$ \phi\left(d(g\varkappa, F(\varkappa, y))\right) \leq \phi\left(d(g\varkappa, F(\varkappa_n, y_n))\right) + \frac{1}{2}\phi\left(d(g\varkappa_n, g\varkappa) + d(gy_n, gy)\right) - \psi\left(\frac{d(g\varkappa_n, g\varkappa) + d(gy_n, gy)}{2}\right), $$

on letting $n \to \infty$, we get

$$ \phi\left(d(g\varkappa, F(\varkappa, y))\right) \leq \lim_{n\to\infty}\phi\left(d(g\varkappa, F(\varkappa_n, y_n))\right) + \frac{1}{2}\lim_{n\to\infty}\phi\left(d(g\varkappa_n, g\varkappa) + d(gy_n, gy)\right) - \lim_{n\to\infty}\psi\left(\frac{d(g\varkappa_n, g\varkappa) + d(gy_n, gy)}{2}\right). \qquad (6.2.46) $$

Using (6.2.38) and properties of ϕ and ψ, it follows that $\phi\left(d(g\varkappa, F(\varkappa, y))\right) = 0$, so that $d(g\varkappa, F(\varkappa, y)) = 0$. Thus, $g\varkappa = F(\varkappa, y)$. Similarly, we can obtain $gy = F(y, \varkappa)$. Hence, F and g have coupled coincidence point in X.

Comparison Of Our Technique With The Already Existing Technique

In their work, Sintunavarat et al. [166] and Hussain et al. [167] requires to prove the results for a single mapping and then extends the obtained results for a pair of mappings to establish the existence of coupled coincidence points. But in our results, we do not require to prove any results for a single mapping followed by extending it to a pair of mappings, rather, we have given a direct proof to obtain the coupled coincidence point results. In order to produce and compare our technique with the technique used by Hussain et al. [167], we have used the same contractive conditions used by Hussain et al. [167].

Case (b) of the Theorems 6.2.1 and 6.2.2 not only relaxes the continuity hypothesis of the mapping F but also relaxes the continuity assumption of the mapping g. But case (b) of Theorems 6.1.1 and 6.1.2 (proved by Hussain et al. [167]), relaxes only the continuity assumption of the mapping F and not of the mapping g.

In view of this discussion, we can conclude that the technique used by us improves the technique of Sintunavarat et al. [166] used by Hussain et al. [167].

Next, we present an example in support of our results:

Example 6.2.1. Consider the POMS $(\mathcal{X}, \preccurlyeq, d)$ where $\mathcal{X} = (0, 1]$, the natural ordering $\leq$ of real numbers as the partial ordering and $d(\varkappa, y) = |\varkappa - y|$ for all x, y in $\mathcal{X}$. Then, $\mathcal{X}$ satisfies Assumption 2.1.7.

Define the mappings $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $g: \mathcal{X} \to \mathcal{X}$ by

$$F(\varkappa, y) = 0.5 \quad \text{and} \quad g\varkappa = \begin{cases} 0.4 & \text{if } 0 < \varkappa < 0.6, \\ \varkappa - 0.3 & \text{if } 0.6 \leq \varkappa \leq 1, \end{cases} \text{ for } \varkappa, y \text{ in } \mathcal{X}.$$

Since $gF(\varkappa, y) = g(0.5) = 0.4 \neq 0.5 = F(g\varkappa, gy)$ for all $\varkappa$, y in $\mathcal{X}$, the mappings F, g are not commutative. Also, the pair (F, g) is not compatible. For, consider the sequences $\{\varkappa_n\} = \left\{0.8 + \frac{1}{n}\right\}$ and $\{y_n\} = \left\{0.8 - \frac{1}{n}\right\}$ for all $n \geq 5$, then

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = 0.5 = \lim_{n\to\infty} g\varkappa_n, \quad \lim_{n\to\infty} F(y_n, \varkappa_n) = 0.5 = \lim_{n\to\infty} gy_n.$$

Then, it follows that

$$\lim_{n\to\infty} d\big(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n)\big) = 0.1 \neq 0,$$

$$\lim_{n\to\infty} d\big(gF(y_n, \varkappa_n), F(gy_n, g\varkappa_n)\big) = 0.1 \neq 0.$$

Therefore, the pair (F, g) is not compatible. Clearly, $F(\mathcal{X} \times \mathcal{X}) = \{0.5\} \subseteq [0.3, 0.7] = g(\mathcal{X})$, g is not continuous, $g(\mathcal{X})$ is complete and F has MgMP.

Also, there exist $\varkappa_0 = 0.2$ and $y_0 = 0.9$ such that $g\varkappa_0 = g(0.2) = 0.4 \leq 0.5 = F(0.2, 0.9) = F(\varkappa_0, y_0)$ and $gy_0 = g(0.9) = 0.6 \geq 0.5 = F(0.9, 0.2) = F(y_0, \varkappa_0)$.

Further, the contractive conditions involved in Theorems 6.2.1 and 6.2.2 also hold due to the choice of F and g. Hence, all the conditions of Theorem 6.2.1 and 6.2.2 are satisfied. Therefore, F and g have a coupled coincidence point in $\mathcal{X}$, which indeed is (0.8, 0.8).

Remark 6.2.1. Theorems 6.1.1 and 6.1.2 cannot be applied to Example 6.2.1 since in Example 6.2.1, g is not continuous but using Theorems 6.2.1 and 6.2.2 we obtained coupled coincidence points under the same contractive conditions as used in Theorems 6.1.1 and 6.1.2, respectively. This shows that Theorems 6.2.1 and 6.2.2 are true generalizations of Theorems 6.1.1 and 6.1.2, respectively.

6.3 IMPROVEMENT OF SOME COUPLED COINCIDENCE POINT RESULTS

In this section, using the technique discussed in Section 6.2, we improve the recent results of Choudhury et al. [56] and Alsulami [168].

Choudhury et al. [56] established the existence of coupled coincidence points under Theorem 2.1.18, which is again stated below (for the sake of convenience):

Theorem 6.3.1 ([56]). Let (Ӽ, ≼, ᶑ) be a POCMS. Let ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that ϕ(ƭ) = 0 iff ƭ = 0 and ψ be an ADF. Let Ғ: Ӽ × Ӽ → Ӽ and ɡ: Ӽ → Ӽ be two mappings such that Ғ has MɡMP on Ӽ and

$$\psi\left(ᶑ\big(Ғ(\varkappa, y), Ғ(u, v)\big)\right) \leq \psi(\max\{ᶑ(ɡ\varkappa, ɡu), ᶑ(ɡy, ɡv)\})$$
$$- \phi(\max\{ᶑ(ɡ\varkappa, ɡu), ᶑ(ɡy, ɡv)\}), \qquad (6.3.1)$$

for all ϰ, y, u, v ∈ Ӽ for which ɡϰ ≽ ɡu, ɡy ≼ ɡv. Suppose that ɡ be continuous, Ғ(Ӽ × Ӽ) ⊆ ɡ(Ӽ) and the pair (Ғ, ɡ) be compatible. Also, suppose that

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 2.1.8.

If Ӽ has the property (P2), then Ғ and ɡ have a coupled coincidence point in Ӽ.

Alsulami [168] obtained coupled coincidence points under contractive condition (6.3.1) by considering ϕ and ψ both to be ADF and replacing the Assumption 2.1.8 by Assumption 2.1.7.

Theorem 6.3.2 ([168]). Let (Ӽ, ≼, ᶑ) be a POCMS. Let ϕ and ψ be two ADF and Ғ: Ӽ × Ӽ → Ӽ, ɡ: Ӽ → Ӽ be two mappings such that Ғ has MɡMP on Ӽ and satisfy (6.3.1) for all ϰ, y, u, v ∈ Ӽ for which ɡϰ ≽ ɡu, ɡy ≼ ɡv. Suppose that Ғ(Ӽ × Ӽ) ⊆ ɡ(Ӽ), ɡ be continuous and monotone increasing and the pair (Ғ, ɡ) is compatible. Suppose either

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ӽ has the property (P2), then Ғ and ɡ have a coupled coincidence point in Ӽ.

Now, using the technique discussed in section 6.2, we improve Theorems 6.3.1 and 6.3.2 as follows:

Theorem 6.3.3. Let (Ӽ, ≼, ᶑ) be a POMS and ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that ϕ(ƭ) = 0 iff ƭ = 0 and ψ be an ADF. Let Ғ: Ӽ × Ӽ → Ӽ and ɡ: Ӽ → Ӽ be two mappings such that Ғ has MɡMP on Ӽ and satisfy (6.3.1) for all ϰ, y, u, v in Ӽ for which ɡϰ ≽ ɡu, ɡy ≼ ɡv. Assume that Ғ(Ӽ × Ӽ) ⊆ ɡ(Ӽ), ɡ(Ӽ) is a complete subspace of Ӽ. Also, assume either

(a) Ғ and ɡ both are continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ӽ has the property (P2), then Ғ and ɡ have a coupled coincidence point in Ӽ.

Proof. As Ӽ has property (P2), there exist $\varkappa_0$, y_0 in Ӽ such that $ɡ\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ and $ɡy_0 \succcurlyeq Ғ(y_0, \varkappa_0)$. Also, as Ғ(Ӽ × Ӽ) ⊆ ɡ(Ӽ) and Ғ has MɡMP on Ӽ, then as in the proof of Theorem 3.2.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in X such that

$$g\varkappa_{n+1} = F(\varkappa_n, y_n),\ gy_{n+1} = F(y_n, \varkappa_n) \text{ for all } n \geq 0 \tag{6.3.2}$$

and
$$g\varkappa_n \preccurlyeq g\varkappa_{n+1}, \quad gy_n \succcurlyeq gy_{n+1} \text{ for all } n \geq 0. \tag{6.3.3}$$

We suppose that either $g\varkappa_{n+1} = F(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = F(y_n, \varkappa_n) \neq gy_n$, for all $n \geq 0$, otherwise, the result follows trivially.

Let $R_n = \max\{d(g\varkappa_n, g\varkappa_{n+1}), d(gy_n, gy_{n+1})\}$. We shall show that

$$\psi(R_n) \leq \psi(R_{n-1}) - \phi(R_{n-1}). \tag{6.3.4}$$

Since $g\varkappa_n \preccurlyeq g\varkappa_{n+1}$, $gy_n \succcurlyeq gy_{n+1}$ for all $n \geq 1$, by (6.3.1) and (6.3.2), we get

$$\begin{aligned}\psi(d(g\varkappa_n, g\varkappa_{n+1})) &= \psi(d(F(\varkappa_{n-1}, y_{n-1}), F(\varkappa_n, y_n)))\\ &\leq \psi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\})\\ &\quad - \phi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\}).\end{aligned} \tag{6.3.5}$$

Similarly, for all $n \geq 1$, we get

$$\begin{aligned}\psi(d(gy_n, gy_{n+1})) &\leq \psi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\})\\ &\quad - \phi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\}).\end{aligned} \tag{6.3.6}$$

By (6.3.5) and (6.3.6) and using monotone property of ψ, we get

$$\begin{aligned}&\psi(\max\{d(g\varkappa_n, g\varkappa_{n+1}), d(gy_n, gy_{n+1})\})\\ &= \max\{\psi(d(g\varkappa_n, g\varkappa_{n+1})), \psi(d(gy_n, gy_{n+1}))\}\\ &\leq \psi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\})\\ &\quad - \phi(\max\{d(g\varkappa_{n-1}, g\varkappa_n), d(gy_{n-1}, gy_n)\}),\end{aligned}$$

so that, (6.3.4) holds. Now, since $\phi(f) > 0$ for $f > 0$, by (6.3.4), for all $n \geq 0$, we have $\psi(R_n) \leq \psi(R_{n-1})$, which implies on using monotone property of ψ, that $R_n \leq R_{n-1}$. Thus, $\{R_n\}$ is a monotone decreasing sequence of non-negative real numbers, so, there exists some $R \geq 0$ such that $\lim_{n\to\infty} R_n = R$. Now, letting $n \to \infty$ in (6.3.4), we get $\psi(R) \leq \psi(R) - \phi(R)$, a contradiction unless $R = 0$. Hence,

$$\lim_{n\to\infty} \max\{d(g\varkappa_n, g\varkappa_{n+1}), d(gy_n, gy_{n+1})\} = \lim_{n\to\infty} R_n = 0, \tag{6.3.7}$$

so that
$$\lim_{n\to\infty} d(g\varkappa_n, g\varkappa_{n+1}) = \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0.$$

We claim that both $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. If possible let at least one of the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{n(k)\}$ such that for every $k \in \mathbb{N}$,

$$n(k) > m(k) \geq k$$

and
$$d_k = \max\{d(g\varkappa_{m(k)}, g\varkappa_{n(k)}), d(gy_{m(k)}, gy_{n(k)})\} \geq \varepsilon. \tag{6.3.8}$$

Now corresponding to m(k) there exists a smallest n(k) ∈ ℕ for which (6.3.8) holds. Then,

$$\max\{d(g\varkappa_{m(k)}, g\varkappa_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})\} < \varepsilon. \quad (6.3.9)$$

Now, $\varepsilon \leq d_k = \max\{d(g\varkappa_{m(k)}, g\varkappa_{n(k)}), d(gy_{m(k)}, gy_{n(k)})\}$

$$\leq \max\{d(g\varkappa_{m(k)}, g\varkappa_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})\}$$

$$+ \max\{d(g\varkappa_{n(k)-1}, g\varkappa_{n(k)}), d(gy_{n(k)-1}, gy_{n(k)})\},$$

that is, $\varepsilon \leq d_k < \varepsilon + R_{n(k)-1}$,

which implies on letting $k \to \infty$ and using (6.3.7), that

$$\lim_{k\to\infty} d_k = \varepsilon. \quad (6.3.10)$$

Further, it also follows easily that

$$\lim_{k\to\infty} d_{k+1} = \varepsilon. \quad (6.3.11)$$

Since n(k) > m(k), $g\varkappa_{n(k)} \succcurlyeq g\varkappa_{m(k)}$ and $gy_{n(k)} \preccurlyeq gy_{m(k)}$. Then, by (6.3.1) and (6.3.2), we get

$$\psi\big(d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})\big) = \psi\big(d(F(\varkappa_{n(k)}, y_{n(k)}), F(\varkappa_{m(k)}, y_{m(k)}))\big)$$

$$\leq \psi\big(\max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}\big)$$

$$- \phi\big(\max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}\big),$$

so that,

$$\psi\big(d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})\big) \leq \psi\big(d_k\big) - \phi\big(d_k\big). \quad (6.3.12)$$

Similarly,

$$\psi\big(d(gy_{n(k)+1}, gy_{m(k)+1})\big) \leq \psi\big(d_k\big) - \phi\big(d_k\big). \quad (6.3.13)$$

By (6.3.12) and (6.3.13) and using the monotone property of ψ, we get

$$\psi\big(d_{k+1}\big) = \psi\big(\max\{d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1})\}\big)$$

$$= \max\{\psi(d(g\varkappa_{n(k)+1}, g\varkappa_{m(k)+1})), \psi(d(gy_{n(k)+1}, gy_{m(k)+1}))\}$$

$$\leq \psi\big(d_k\big) - \phi\big(d_k\big). \quad (6.3.14)$$

Letting $k \to \infty$ in (6.3.14), using (6.3.10), (6.3.11) and the continuity of ψ and ϕ, we obtain that $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$, a contradiction. Hence, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences in X and hence in g(X). As g(X) is complete, there exist some $\varkappa$, y in X such that

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \quad \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \quad (6.3.15)$$

We finally show that $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$.

Now, by Lemma 6.1.1, there exists some $A \subseteq X$ such that $g(A) = g(X)$ and the mapping $g: A \to X$ is one-to-one. Define a mapping $H: g(A) \times g(A) \to X$ by

$$H(ga, gb) = F(a, b) \text{ for } ga, gb \in g(A). \tag{6.3.16}$$

Since g is one-to-one on A, so H is well defined.

By (6.3.15) and (6.3.16), we have

$$\lim_{n\to\infty} H(g\varkappa_n, gy_n) = \lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \tag{6.3.17}$$

and $$\lim_{n\to\infty} H(gy_n, g\varkappa_n) = \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \tag{6.3.18}$$

Let the assumption (a) holds. As F and g both are continuous, H is also continuous. Then, by (6.3.17) and (6.3.18), we get

$$H(g\varkappa, gy) = g\varkappa \text{ and } H(gy, g\varkappa) = gy. \tag{6.3.19}$$

By (6.3.16) and (6.3.19), we get $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

Now, let the assumption (b) holds. By (6.3.3), (6.3.17) and (6.3.18), we get that $\{g\varkappa_n\}$ is a non-decreasing sequence converging to $g\varkappa$ and $\{gy_n\}$ is a non-increasing sequence converging to gy. Therefore, by assumption, for all $n \geq 0$, we have

$$g\varkappa_n \preccurlyeq g\varkappa \text{ and } gy \preccurlyeq gy_n. \tag{6.3.20}$$

We suppose that $(g\varkappa_n, gy_n) \neq (g\varkappa, gy)$ for all $n \geq 0$, otherwise, the result follows trivially. Now, using (6.3.1) for (6.3.20), we get

$$\psi\left(d\big(F(\varkappa, y), F(\varkappa_n, y_n)\big)\right) \leq \psi(\max\{d(g\varkappa, g\varkappa_n), d(gy, gy_n)\}) - \phi(\max\{d(g\varkappa, g\varkappa_n), d(gy, gy_n)\}). \tag{6.3.21}$$

By triangle inequality and monotone property of ψ, we get

$$\psi\left(d\big(g\varkappa, F(\varkappa, y)\big)\right) \leq \psi(d(g\varkappa, g\varkappa_{n+1}) + d(g\varkappa_{n+1}, F(\varkappa, y))) = \psi(d(g\varkappa, g\varkappa_{n+1}) + d(F(\varkappa_n, y_n), F(\varkappa, y))). \tag{6.3.22}$$

Letting $n \to \infty$ in (6.3.22), we have

$$\psi\left(d\big(g\varkappa, F(\varkappa, y)\big)\right) \leq \lim_{n\to\infty} \psi(d(g\varkappa, g\varkappa_{n+1}) + d(F(\varkappa_n, y_n), F(\varkappa, y))).$$

Now, by continuity of ψ and (6.3.15), we get

$$\psi\left(d\big(g\varkappa, F(\varkappa, y)\big)\right) \leq \lim_{n\to\infty} \psi(d(F(\varkappa_n, y_n), F(\varkappa, y))). \tag{6.3.23}$$

Inserting (6.3.21) in (6.3.23), we get

$$\psi\left(d\big(g\varkappa, F(\varkappa, y)\big)\right) \leq \lim_{n\to\infty} [\psi(\max\{d(g\varkappa, g\varkappa_n), d(gy, gy_n)\}) - \phi(\max\{d(g\varkappa, g\varkappa_n), d(gy, gy_n)\})].$$

Using (6.3.15) and the properties of ψ, ϕ, we get $d(g\varkappa, F(\varkappa, y)) = 0$, so that $g\varkappa = F(\varkappa, y)$. Similarly, we can get $gy = F(y, \varkappa)$.

Hence, in both the cases, Ғ and g have a coupled coincidence point.

Remark 6.3.1. (i) Though, the contraction condition used in Theorems 6.3.3 and 6.3.1 are same but in Theorem 6.3.3, we do not require the pair of compatible mappings and also, the completeness of the space Ӽ has been relaxed by assuming the completeness of the range space of the mapping g. Further, Case (b) of Theorem 6.3.3 not only relaxes the continuity assumption of the mapping Ғ but also relaxes the continuity hypothesis of the mapping g which has not been relaxed in Case (b) of Theorem 6.3.1.

(ii) The above comparison between Theorems 6.3.3 and 6.3.1 is also valid between Theorems 6.3.3 and 6.3.2, respectively. Further, the mapping ϕ is an ADF in Theorem 6.3.2 but in Theorem 6.3.3 the monotone increasing assumption of ϕ has also been relaxed. Finally, Theorem 6.3.3 does not require the monotone increasing assumption of the mapping g which has been considered in Theorem 6.3.2.

Hence, we can conclude that Theorem 6.3.3 improves Theorems 6.3.1 and 6.3.2.

6.4 GENERALIZATION OF A COUPLED COINCIDENCE POINT RESULT IN MENGER PM-SPACES

In this section, using the technique discussed in section 6.2, we improve the recent result of Choudhury et al. [119] in POCMPMS.

Recently, Fang [114] introduced the following class of gauge function and utilized it to obtain some results in PM-spaces:

Definition 6.4.1 ([114]). Let Ω_W denote the class of all functions φ: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition: "for each ƫ > 0, there exists ɍ ≥ ƫ such that $\lim_{n\to\infty}\varphi^n(ɍ) = 0$".

Lemma 6.4.1 ([114]). "Let $\varphi \in \Omega_W$, then, for each ƫ > 0, there exists ɍ ≥ ƫ such that $\varphi(ɍ) <$ ƫ ".

Using the gauge function φ, Choudhury et al. [119] proved the following result:

Theorem 6.4.1 ([119]). Let (Ӽ, ≼, F, Δ) be a POCMPMS, where Δ is a continuous Hadžić type t-norm. Let g: Ӽ → Ӽ and Q: Ӽ × Ӽ → Ӽ be two mappings such that Q has MgMP. Let there exists $\varphi \in \Omega_W$ such that

$$F_{Q(\varkappa,y),Q(u,v)}\big(\varphi(ƫ)\big) \geq \left[F_{g\varkappa,gu}(ƫ).F_{gy,gv}(ƫ)\right]^{\frac{1}{2}}, \qquad (6.4.1)$$

for all ƫ > 0 and ϰ, y, u, v in Ӽ with gϰ ≼ gu and gy ≽ gv. Let g be monotone increasing and continuous, Q(Ӽ × Ӽ) ⊆ g(Ӽ) and the pair (g, Q) is compatible. Also, suppose either

(a) Q is continuous, or (b) $\mathfrak{X}$ assumes Assumption 2.1.7.

If $\mathfrak{X}$ has the property **(P2)** w.r.t. g and Q, that is, "there exist $\varkappa_0$, $y_0 \in \mathfrak{X}$ such that $g\varkappa_0 \preccurlyeq Q(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq Q(y_0, \varkappa_0)$". Then, Q and g have a coupled coincidence point in $\mathfrak{X}$.

We now generalize Theorem 6.4.1 as follows:

Theorem 6.4.2. Let ($\mathfrak{X}$, $\preccurlyeq$, F, Δ) be a POMPMS, where Δ is a continuous Hadžić type t-norm. Let g: $\mathfrak{X} \to \mathfrak{X}$ and Q: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be two mappings such that Q has MgMP. Let there exists $\varphi \in \Omega_W$ such that (6.4.1) holds for all $\mathfrak{f} > 0$ and $\varkappa$, y, u, v in $\mathfrak{X}$ with $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$. Let $Q(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$ and one of $Q(\mathfrak{X} \times \mathfrak{X})$ or $g(\mathfrak{X})$ is a complete subspace of $\mathfrak{X}$. Also, assume either

(a) g and Q both are continuous, or (b) $\mathfrak{X}$ assumes Assumption 2.1.7.

If $\mathfrak{X}$ has the property (P2) w.r.t. g and Q, then Q and g have a coupled coincidence point in $\mathfrak{X}$.

Proof. As $\mathfrak{X}$ has property (P2) w.r.t. g and Q, there exist $\varkappa_0$, y_0 in $\mathfrak{X}$ such that $g\varkappa_0 \preccurlyeq Q(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq Q(y_0, \varkappa_0)$. Also, as $Q(\mathfrak{X} \times \mathfrak{X}) \subseteq g(\mathfrak{X})$ and Q has MgMP on $\mathfrak{X}$, then, as in proof of Theorem 3.2.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ can be constructed in $\mathfrak{X}$ such that

$$g\varkappa_{n+1} = Q(\varkappa_n, y_n),\ gy_{n+1} = Q(y_n, \varkappa_n) \text{ for all } n \geq 0, \tag{6.4.2}$$

and

$$g\varkappa_n \preccurlyeq g\varkappa_{n+1},\ gy_n \succcurlyeq gy_{n+1} \text{ for all } n \geq 0. \tag{6.4.3}$$

We suppose that either $g\varkappa_{n+1} = Q(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = Q(y_n, \varkappa_n) \neq gy_n$, otherwise, the result follows trivially.

Now, for $\mathfrak{f} > 0$, and $n \geq 1$, by (6.4.1) − (6.4.3), we get

$$F_{g\varkappa_n,\, g\varkappa_{n+1}}\big(\varphi(\mathfrak{f})\big) = F_{Q(\varkappa_{n-1}, y_{n-1}), Q(\varkappa_n, y_n)}\big(\varphi(\mathfrak{f})\big)$$

$$\geq \left[F_{g\varkappa_{n-1}, g\varkappa_n}(\mathfrak{f}).\, F_{gy_{n-1}, gy_n}(\mathfrak{f})\right]^{\frac{1}{2}}. \tag{6.4.4}$$

Similarly, for $\mathfrak{f} > 0$, we get

$$F_{gy_n,\, gy_{n+1}}\big(\varphi(\mathfrak{f})\big) \geq \left[F_{g\varkappa_{n-1}, g\varkappa_n}(\mathfrak{f}).\, F_{gy_{n-1}, gy_n}(\mathfrak{f})\right]^{\frac{1}{2}}. \tag{6.4.5}$$

Let

$$A_n(\mathfrak{f}) = \left[F_{g\varkappa_{n-1}, g\varkappa_n}(\mathfrak{f}).\, F_{gy_{n-1}, gy_n}(\mathfrak{f})\right]^{\frac{1}{2}}. \tag{6.4.6}$$

Then, by (6.4.4) and (6.4.5), we get

$$F_{g\varkappa_n,\, g\varkappa_{n+1}}\big(\varphi(\mathfrak{f})\big) \,.\, F_{gy_n,\, gy_{n+1}}\big(\varphi(\mathfrak{f})\big) \geq A_n(\mathfrak{f}).A_n(\mathfrak{f}),$$

which implies that

$$\left[A_{n+1}\big(\varphi(\mathfrak{f})\big)\right]^2 \geq [A_n(\mathfrak{f})]^2,$$

so that $\left[A_{n+1}(\varphi(\mathfrak{t}))\right] \geq [A_n(\mathfrak{t})]$. (6.4.7)

On repeatedly applying (6.4.7), using (6.4.4) and (6.4.5), respectively, for all $\mathfrak{t} > 0$, n > 1, we get

$$F_{g\varkappa_n, g\varkappa_{n+1}}(\varphi^n(\mathfrak{t})) \geq A_n(\varphi^{n-1}(\mathfrak{t})) \geq \ldots \geq A_1(\mathfrak{t}) = \left[F_{g\varkappa_0, g\varkappa_1}(\mathfrak{t}). F_{gy_0, gy_1}(\mathfrak{t})\right]^{\frac{1}{2}},$$

and $F_{gy_n, gy_{n+1}}(\varphi^n(\mathfrak{t})) \geq A_n(\varphi^{n-1}(\mathfrak{t})) \geq \ldots \geq A_1(\mathfrak{t}) = \left[F_{g\varkappa_0, g\varkappa_1}(\mathfrak{t}). F_{gy_0, gy_1}(\mathfrak{t})\right]^{\frac{1}{2}}$.

Now, for $\mathfrak{t} > 0$, we shall show

$$\lim_{n\to\infty} F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t}) = 1 \quad \text{and} \quad \lim_{n\to\infty} F_{gy_n, gy_{n+1}}(\mathfrak{t}) = 1. \tag{6.4.8}$$

Since $F_{g\varkappa_0, g\varkappa_1}(\mathfrak{t}) \to 1$, $F_{gy_0, gy_1}(\mathfrak{t}) \to 1$ as $\mathfrak{t} \to \infty$, for $\varepsilon \in (0, 1]$ there exists $\mathfrak{t}_1 > 0$ such that $F_{g\varkappa_0, g\varkappa_1}(\mathfrak{t}_1) > 1 - \varepsilon$ and $F_{gy_0, gy_1}(\mathfrak{t}_1) > 1 - \varepsilon$. Also, $\varphi \in \Omega_W$, so there exist $\mathfrak{t}_0 \geq \mathfrak{t}_1$ such that $\lim_{n\to\infty} \varphi^n(\mathfrak{t}_0) = 0$. Therefore, to each $\mathfrak{t} > 0$, there exists $n_0 \geq 1$ such that $\varphi^n(\mathfrak{t}_0) < \mathfrak{t}$ for all $n \geq n_0$. Then, for all $n \geq n_0$, we have

$$\begin{aligned} F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t}) &\geq F_{g\varkappa_n, g\varkappa_{n+1}}(\varphi^n(\mathfrak{t}_0)) \\ &\geq \left[F_{g\varkappa_0, g\varkappa_1}(\mathfrak{t}). F_{gy_0, gy_1}(\mathfrak{t})\right]^{\frac{1}{2}} \\ &> \left[(1-\varepsilon).(1-\varepsilon)\right]^{\frac{1}{2}} = (1-\varepsilon). \end{aligned}$$

Hence, for $\mathfrak{t} > 0$, $\lim_{n\to\infty} F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t}) = 1$. Similarly, we can get $\lim_{n\to\infty} F_{gy_n, gy_{n+1}}(\mathfrak{t}) = 1$. Therefore, (6.4.8) holds. Now, by (6.4.6), we have

$$A_n(\mathfrak{t}) \to 1 \text{ as } n \to \infty. \tag{6.4.9}$$

Since $\varphi \in \Omega_W$, by Lemma 6.4.1, for $\mathfrak{t} > 0$, there exists $\mathfrak{r} \geq \mathfrak{t}$ such that $\varphi(\mathfrak{r}) < \mathfrak{t}$. Let $n \geq 1$ be given. Now, by induction, for $\Bbbk \geq 1$, we shall show

$$F_{g\varkappa_n, g\varkappa_{n+\Bbbk}}(\mathfrak{t}) \geq \Delta^{\Bbbk-1}(A_n(\mathfrak{t} - \varphi(\mathfrak{r}))) \text{ and } F_{gy_n, gy_{n+\Bbbk}}(\mathfrak{t}) \geq \Delta^{\Bbbk-1}(A_n(\mathfrak{t} - \varphi(\mathfrak{r}))). \tag{6.4.10}$$

Since $\Delta^0(\mathfrak{t}) = \mathfrak{t}$, therefore, (6.4.10) is true for $\Bbbk = 1$. Let (6.4.10) is true for some $\Bbbk$. Then,

$$F_{g\varkappa_n, g\varkappa_{n+\Bbbk+1}}(\mathfrak{t}) = F_{g\varkappa_n, g\varkappa_{n+\Bbbk+1}}(\mathfrak{t} - \varphi(\mathfrak{r}) + \varphi(\mathfrak{r}))$$

$$\geq \Delta\left(F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t} - \varphi(\mathfrak{r})), F_{g\varkappa_{n+1}, g\varkappa_{n+\Bbbk+1}}(\varphi(\mathfrak{r}))\right)$$

$$\geq \Delta\left(F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t} - \varphi(\mathfrak{r})), \left[F_{g\varkappa_n, g\varkappa_{n+\Bbbk}}(\mathfrak{r}). F_{gy_n, gy_{n+\Bbbk}}(\mathfrak{r})\right]^{\frac{1}{2}}\right) \quad \text{(by (6.4.1) and (6.4.3))}$$

$$\geq \Delta\left(F_{g\varkappa_n, g\varkappa_{n+1}}(\mathfrak{t} - \varphi(\mathfrak{r})), \left[F_{g\varkappa_n, g\varkappa_{n+\Bbbk}}(\mathfrak{t}). F_{gy_n, gy_{n+\Bbbk}}(\mathfrak{t})\right]^{\frac{1}{2}}\right) \quad \text{(since } \mathfrak{r} \geq \mathfrak{t}\text{)}$$

$$\geq \Delta\left(A_n(\mathfrak{t} - \varphi(\mathfrak{r})), \left[\Delta^{\Bbbk-1}(A_n(\mathfrak{t} - \varphi(\mathfrak{r}))). \Delta^{\Bbbk-1}(A_n(\mathfrak{t} - \varphi(\mathfrak{r})))\right]^{\frac{1}{2}}\right)$$

$\geq \Delta\big(A_n\big(t - \varphi(r)\big), \Delta^{k-1}(A_n(t - \varphi(r)))\big) = \Delta^{k}(A_n(t - \varphi(r)))$.

Similarly, we have $F_{gy_n, gy_{n+k+1}}(t) \geq \Delta^{k}(A_n(t - \varphi(r)))$. Therefore, by induction, (6.4.10) holds for all $k \geq 1$ and $t > 0$. Now, we prove that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since, Δ is a Hadžić type t-norm, the family of iterates $\{\Delta^{p}\}$ is equi-continuous at the point s = 1, that is, there exists $\delta \in (0, 1)$ such that

$$\Delta^{p}(s) > 1 - \delta, \qquad (6.4.11)$$

whenever $1 \geq s > 1 - \varepsilon$ and $p \geq 1$. By (6.4.9), there exists some $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have

$$A_n\big(t - \varphi(r)\big) > 1 - \varepsilon. \qquad (6.4.12)$$

Then, for all $n \geq n_0$, $k \geq 1$, it follows from (6.4.10), (6.4.11) and (6.4.12) that

$$F_{g\varkappa_n, g\varkappa_{n+k}}(t) \geq \Delta^{k-1}(A_n(t - \varphi(r))) > 1 - \varepsilon \quad \text{and} \quad F_{gy_n, gy_{n+k}}(t) > 1 - \varepsilon. \qquad (6.4.13)$$

Now, (6.4.13) implies that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences.

W.L.O.G., assume that $g(X)$ is complete, so there exist $\varkappa$, y in X such that

$$\lim_{n\to\infty} Q(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \quad \lim_{n\to\infty} Q(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \qquad (6.4.14)$$

We now show that $g\varkappa = Q(\varkappa, y)$ and $gy = Q(y, \varkappa)$.

Now, by Lemma 6.1.1, there exists a subset $A \subseteq X$ such that $g(A) = g(X)$ and the mapping g is one-to-one on A. Let us define a mapping $H: g(A) \times g(A) \to X$ by

$$H(ga, gb) = Q(a, b) \qquad (6.4.15)$$

for all $ga, gb \in g(A)$. As g is one-to-one on A, so H is well-defined. By (6.4.14) and (6.4.15), we get

$$\lim_{n\to\infty} H(g\varkappa_n, gy_n) = \lim_{n\to\infty} Q(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \qquad (6.4.16)$$

$$\lim_{n\to\infty} H(gy_n, g\varkappa_n) = \lim_{n\to\infty} Q(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \qquad (6.4.17)$$

Let the assumption (a) holds.

As both Q and g are continuous, so H is also continuous. Then, by (6.4.16) and (6.4.17), we get

$$H(g\varkappa, gy) = g\varkappa \text{ and } H(gy, g\varkappa) = gy. \qquad (6.4.18)$$

By (6.4.15) and (6.4.18), we get $Q(\varkappa, y) = g\varkappa$ and $Q(y, \varkappa) = gy$.

Now, suppose assumption (b) holds.

By (6.4.3) and (6.4.14), $\{g\varkappa_n\}$ is a non-decreasing sequence converging to $g\varkappa$ and $\{gy_n\}$ is a non-increasing sequence converging to gy. Hence, by assumption, for all $n \geq 0$, we get

$$g\varkappa_n \preccurlyeq g\varkappa \text{ and } gy \preccurlyeq gy_n \text{ for all } n \geq 0. \qquad (6.4.19)$$

We suppose either $g\varkappa_{n+1} = Q(\varkappa_n, y_n) \neq g\varkappa_n$ or $gy_{n+1} = Q(y_n, \varkappa_n) \neq gy_n$, otherwise we obtain directly the coupled coincidence point of Q and g.

By Lemma 6.4.1, for $\mathfrak{t} > 0$, there exists $\mathfrak{r} \geq \mathfrak{t}$ such that $\phi(\mathfrak{r}) < \mathfrak{t}$.

Now, using (6.4.1) and (6.4.19), we get

$$F_{g\varkappa_{n+1},Q(\varkappa,y)}(\mathfrak{t}) \geq F_{g\varkappa_{n+1},Q(\varkappa,y)}\big(\phi(\mathfrak{r})\big) = F_{Q(\varkappa_n,y_n),Q(\varkappa,y)}\big(\phi(\mathfrak{r})\big)$$

$$\geq \left[F_{g\varkappa_n,\, g\varkappa}(\mathfrak{r}).\, F_{gy_n,\, gy}(\mathfrak{r})\right]^{\frac{1}{2}}. \qquad (6.4.20)$$

Letting $n \to \infty$ in (6.4.20) and using (6.4.16) and (6.4.17), we get $F_{g\varkappa,\, Q(\varkappa,y)}(\mathfrak{t}) \geq 1$, hence, we get $F_{g\varkappa,\, Q(\varkappa,y)}(\mathfrak{t}) = 1$ for all $\mathfrak{t} > 0$, so that $g\varkappa = Q(\varkappa, y)$. Similarly, we can get $gy = Q(y, \varkappa)$.

Therefore, in both the cases g and Q have a coupled coincidence point in Ӿ.

Remark 6.4.1. Though, the contraction used in Theorems 6.4.1 and 6.4.2 are same but in Theorem 6.4.2, we do not require the pair of compatible mappings and also, the completeness of the space Ӿ has been replaced by the completeness of g(Ӿ). Also, in Theorem 6.4.2, the mapping g is not monotone increasing. Further, Case (b) of Theorem 6.4.2 not only relaxes the continuity assumption of the mapping Q but also relaxes the continuity hypothesis of the mapping g which has not been relaxed in Case (b) of Theorem 6.4.1.

In view of Remark 6.4.1, we can conclude that Theorem 6.4.2 improves Theorem 6.4.1.

6.5 IMPROVEMENT OF A COUPLED COINCIDENCE POINT RESULT IN G-METRIC SPACES

In this section, using the technique discussed in section 6.2, we generalize Theorem 5.3.1.

Recall that, as in Definition 5.1.1, let Ξ denote the class of functions $\wp: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying: "$\lim_{(\mathfrak{t}_1,\mathfrak{t}_2)\to(\mathfrak{r}_1,\mathfrak{r}_2)} \wp(\mathfrak{t}_1, \mathfrak{t}_2) > 0$ for all $(\mathfrak{r}_1, \mathfrak{r}_2) \in (\mathbb{R}^+)^2$ with $\mathfrak{r}_1 + \mathfrak{r}_2 > 0$".

Theorem 6.5.1. Let (Ӿ, ≼, G) be a POGMS and F: Ӿ × Ӿ → Ӿ, g: Ӿ → Ӿ be the mappings. Suppose there exist some $\wp \in \Xi$ and an ADF π such that for all l, m, n, u, v, w ∈ Ӿ with $gw \preccurlyeq gu \preccurlyeq gl$ and $gm \preccurlyeq gv \preccurlyeq gn$, we have

$$\pi\left(\frac{G\big(F(l,m), F(u,v), F(w,n)\big) + G\big(F(m,l), F(v,u), F(n,w)\big)}{2}\right)$$

$$\leq \pi\left(\frac{G(gl, gu, gw) + G(gm, gv, gn)}{2}\right) - \wp\big(G(gl, gu, gw), G(gm, gv, gn)\big). \qquad (6.5.1)$$

Assume $Ꞙ(Ҳ \times Ҳ) \subseteq g(Ҳ)$, Ꞙ has MgMP and $(g(Ҳ), G)$ or $(Ꞙ(Ҳ \times Ҳ), G)$ is complete. Suppose either

(a) Ꞙ and g both are continuous, or (b) Ҳ assumes Assumption 2.1.7.

If Ҳ has the property (P2), then, Ꞙ and g have a coupled coincidence point in Ҳ.

Proof: As in the proof of Theorem 5.3.1, the sequences $\{g\varkappa_n\}$ and $\{gy_n\}$ defined as under are Cauchy sequences:

$$g\varkappa_n = Ꞙ(\varkappa_{n-1}, y_{n-1}) \text{ and } gy_n = Ꞙ(y_{n-1}, \varkappa_{n-1}), \text{ for all } n \geq 0, \tag{6.5.2}$$

and
$$g\varkappa_n \preccurlyeq g\varkappa_{n+1} \text{ and } gy_n \succcurlyeq gy_{n+1}. \tag{6.5.3}$$

W.L.O.G., let $(g(Ҳ), G)$ be complete, then, there exist $\varkappa$, y in Ҳ such that $\{g\varkappa_n\}$ converges to $g\varkappa$ and $\{gy_n\}$ converges to gy. Then, by Proposition 2.3.3, we get

$$\lim_{n\to\infty} G(g\varkappa_n, g\varkappa_n, \varkappa) = \lim_{n\to\infty}\lim_{n\to\infty} G(g\varkappa_n, \varkappa, \varkappa) = 0, \tag{6.5.4}$$

and
$$\lim_{n\to\infty} G(gy_n, gy_n, y) = \lim_{n\to\infty} G(gy_n, y, y) = 0. \tag{6.5.5}$$

Let assumption (a) holds.

Now, using Lemma 6.1.1, there exists $A \subseteq Ҳ$ such that $g(A) = g(Ҳ)$ and $g: A \to Ҳ$ is one-to-one function. Define the mapping $Ή: g(A) \times g(A) \to Ҳ$ by

$$Ή(g\text{ɑ}, g\text{ƀ}) = Ꞙ(\text{ɑ}, \text{ƀ}) \tag{6.5.6}$$

for all gɑ, gƀ in g(A). As g is one-to-one on A, so Ή is well-defined. Now, by (6.5.4), (6.5.5) and (6.5.6), we get

$$\lim_{n\to\infty} Ή(g\varkappa_n, gy_n) = \lim_{n\to\infty} Ꞙ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \tag{6.5.7}$$

$$\lim_{n\to\infty} Ή(gy_n, g\varkappa_n) = \lim_{n\to\infty} Ꞙ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \tag{6.5.8}$$

As Ꞙ and g are continuous, so Ή is also continuous. Now, using (6.5.7), (6.5.8) and the continuity of Ή, we get

$$Ή(g\varkappa, gy) = g\varkappa \text{ and } Ή(gy, g\varkappa) = gy. \tag{6.5.9}$$

Using (6.5.6) and (6.5.9), we get

$$Ꞙ(\varkappa, y) = g\varkappa \text{ and } Ꞙ(y, \varkappa) = gy.$$

Now, let assumption (b) holds.

Since the non-decreasing sequence $\{g\varkappa_n\}$ converges to $g\varkappa$ and the non-increasing sequence $\{gy_n\}$ converges to gy, by assumption, we get

$$g\varkappa_n \preccurlyeq g\varkappa \text{ and } gy \preccurlyeq gy_n \text{ for } n \geq 0.$$

Then, by (6.5.1), we get

$$\pi\left(\frac{G(Ꞙ(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1}) + G(Ꞙ(y,\varkappa), gy_{n+1}, gy_{n+1})}{2}\right)$$

$$= \pi\left(\frac{G\big(Ꞙ(\varkappa,y), Ꞙ(\varkappa_n,y_n), Ꞙ(\varkappa_n,y_n)\big) + G\big(Ꞙ(y,\varkappa), Ꞙ(y_n,\varkappa_n), Ꞙ(y_n,\varkappa_n)\big)}{2}\right)$$

$$\leq \pi\left(\frac{G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)}{2}\right) - \wp\big(G(g\varkappa, g\varkappa_n, g\varkappa_n), G(gy, gy_n, gy_n)\big)$$
$$\leq \pi\left(\frac{G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)}{2}\right).$$

Now, using the properties of π, we obtain that

$$\frac{G(F(\varkappa,y), g\varkappa_{n+1}, g\varkappa_{n+1}) + G(F(y,\varkappa), gy_{n+1}, gy_{n+1})}{2} \leq \frac{G(g\varkappa, g\varkappa_n, g\varkappa_n) + G(gy, gy_n, gy_n)}{2}. \qquad (6.5.10)$$

Letting $n \to \infty$ (6.5.10), we get $G(F(\varkappa, y), g\varkappa, g\varkappa) \leq 0$ and $G(F(y, \varkappa), gy, gy) \leq 0$, which implies $G(F(\varkappa, y), g\varkappa, g\varkappa) = 0$ and $G(F(y, \varkappa), gy, gy) = 0$, so that $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

Remark 6.5.1. Though, the contraction used in Theorems 5.3.1 and 6.5.1 is same but in Theorem 6.5.1, we do not require the pair of commuting mappings and also, the completeness of the space X has been replaced by assuming the completeness of the range space of any one of the mapping g or F. Also, case (b) of Theorem 6.5.1 relaxes the continuity hypothesis of both the mappings F and g.

Hence, Theorem 6.5.1 generalizes Theorem 5.3.1.

6.6 REMARKS ON SOME RECENT PAPERS CONCERNING COUPLED COINCIDENCE POINTS

In this section, we rectify some gaps and omissions in the works of Alotaibi and Alsulami [68], Turkoglu and Sangurlu [169].

Alotaibi and Alsulami [68] established the existence of coupled coincidence points under Theorem 2.1.20, which is again stated below (for the sake of convenience):

Theorem 6.6.1 ([68]). "Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed g-monotone property on X such that there exist two elements $\varkappa_0, y_0 \in X$ with $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Suppose there exist $\varphi \in \Phi_1$ and $\psi \in \Psi$ such that

$$\varphi\Big(d\big(F(\varkappa, y), F(u,v)\big)\Big) \leq \frac{1}{2}\varphi\big(d(g\varkappa, gu) + d(gy, gv)\big) - \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right), \qquad (6.6.1)$$

for all $\varkappa$, y, u, $v \in X$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and compatible with F and also suppose either

(a) F is continuous, or (b) X has the following property:

(i) if a non-decreasing sequence $\{\varkappa_n\} \to \varkappa$, then $\varkappa_n \preccurlyeq \varkappa$, for all n;

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n.

Then, there exist $\varkappa$, $y \in X$ such that $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$".

Remark 6.6.1. The proof of Theorem 6.6.1 (for proof see [68], Theorem 3.1, page 7, line 12) uses the fact that g is monotone increasing. The hypotheses of this theorem must also include this fact. Also, the statement of Theorem 6.6.1 must include the mapping 'g' which is missing. The correct statement of Theorem 6.6.1 should now read as follows:

Theorem 6.6.2. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ and $g: X \to X$ be two mappings such that F has the mixed g-monotone property on X and there exist two elements $\varkappa_0$, $y_0 \in X$ with $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$. Suppose there exist $\varphi \in \Phi_1$ and $\psi \in \Psi$ such that

$$\varphi\left(d\big(F(\varkappa, y), F(u, v)\big)\right) \leq \frac{1}{2}\varphi\big(d(g\varkappa, gu) + d(gy, gv)\big) - \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right),$$

for all $\varkappa$, y, u, $v \in X$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous, monotone increasing and compatible with F and also suppose either

(a) F is continuous, or (b) X has the following property:

(i) if a non-decreasing sequence $\{\varkappa_n\} \to \varkappa$, then $\varkappa_n \preccurlyeq \varkappa$, for all n;

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n.

Then, there exist $\varkappa$, $y \in X$ such that $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

Afterwards, Turkoglu and Sangurlu [169] using the approach of Hussain et al. [167] established the existence of coupled coincidence points using contraction (6.3.1) under the following result:

Theorem 6.6.3 ([169]). "Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is complete metric space. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X and there exists two elements $\varkappa_0$, y_0 in X such that $\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $y_0 \succcurlyeq F(\varkappa_0, y_0)$. Suppose that F, g satisfy

$$\varphi\left(d\big(F(\varkappa, y), F(u, v)\big)\right) \leq \frac{1}{2}\varphi\big(d(g\varkappa, gu) + d(gy, gv)\big) - \psi\left(\frac{d(g\varkappa, gu) + d(gy, gv)}{2}\right),$$

for all $\varkappa$, y, u, $v \in X$ with $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$, $F(X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous.

Suppose that either

(a) Ꞙ is continuous, or (b) Ϫ has the following property:

(i) if a non-decreasing sequence $\{\varkappa_n\} \to \varkappa$, then $\varkappa_n \preccurlyeq \varkappa$, for all n;

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n.

Then, there exist $\varkappa$, y ∈ Ϫ such that $g\varkappa$ = Ꞙ($\varkappa$, y) and gy = Ꞙ(y, $\varkappa$)".

Remark 6.6.2. (i) It is a well known fact that the coupled coincidence point results for the mappings Ꞙ: Ϫ × Ϫ → Ϫ and g: Ϫ → Ϫ uses the hypotheses that $g\varkappa_0 \preccurlyeq$ Ꞙ($\varkappa_0$, y_0), $gy_0 \succcurlyeq$ Ꞙ(y_0, $\varkappa_0$) and Ꞙ must have mixed g-monotone property. However, the statement of Theorem 6.6.3 includes the incorrect hypotheses that "$\varkappa_0 \preccurlyeq$ Ꞙ($\varkappa_0$, y_0), $y_0 \succcurlyeq$ Ꞙ(y_0, $\varkappa_0$)" and "the mapping Ꞙ has the mixed monotone property". The hypotheses of this result must include the correct facts.

(ii) Further, the statement of Theorem 6.6.3 includes the completeness of the space Ϫ as well as the completeness of the range subspace g(Ϫ). However, the approach used in [169] for proving Theorem 6.6.3 only requires the completeness of the range subspace g(Ϫ) and not the completeness of the space (Ϫ, ꝱ).

(iii) Finally, the statement of Theorem 6.6.3 has repeatedly used the hypothesis: "Let (Ϫ, ≼) be a partially ordered set and suppose there exists a metric ꝱ on Ϫ such that (Ϫ, ꝱ) is complete metric space". This must be corrected. The statement of Theorem 6.6.3 should be corrected and read as follows:

Theorem 6.6.4. Let (Ϫ, ≼) be a partially ordered set and suppose there exists a metric ꝱ on Ϫ such that (Ϫ, ꝱ) is a metric space. Let Ꞙ: Ϫ × Ϫ → Ϫ and g: Ϫ → Ϫ be two mappings such that Ꞙ has the mixed g-monotone property on Ϫ and there exists two elements $\varkappa_0$, y_0 in Ϫ such that $g\varkappa_0 \preccurlyeq$ Ꞙ($\varkappa_0$, y_0) and $gy_0 \succcurlyeq$ Ꞙ($\varkappa_0$, y_0). Suppose there exist $\varphi \in \Phi_1$, $\psi \in \Psi$ such that

$$\varphi\left(ꝱ\left(Ꞙ(\varkappa, y), Ꞙ(u,v)\right)\right) \leq \frac{1}{2}\varphi\left(ꝱ(g\varkappa, gu) + ꝱ(gy, gv)\right) - \psi\left(\frac{ꝱ(g\varkappa, gu)+ꝱ(gy, gv)}{2}\right),$$

for all $\varkappa$, y, u, v ∈ Ϫ with $g\varkappa \preccurlyeq gu$ and $gy \succcurlyeq gv$, Ꞙ(Ϫ × Ϫ) ⊆ g(Ϫ), g(Ϫ) is complete and g is continuous. Suppose that either

(a) Ꞙ is continuous, or (b) Ϫ has the following property:

(i) if a non-decreasing sequence $\{\varkappa_n\} \to \varkappa$, then $\varkappa_n \preccurlyeq \varkappa$, for all n;

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$ for all n.

Then, there exist $\varkappa$, y ∈ Ϫ such that $g\varkappa$ = Ꞙ($\varkappa$, y) and gy = Ꞙ(y, $\varkappa$).

Remark 6.6.3. We note that Theorem 6.6.4 is actually Theorem 6.1.2.

In their work, Alotaibi and Alsulami [68] also claimed to establish the uniqueness of coupled coincidence points obtained under the hypotheses of Theorem 6.6.1 by adding the additional assumption, Assumption 3.3.1, which is again given below (for convenience):

Assumption 3.3.1 ([41, 55]). "For every ($\varkappa$, y), (ζ, t) in $X \times X$, there exists a (u, v) in $X \times X$ that is comparable to ($\varkappa$, y) and (ζ, t)".

To obtain the uniqueness of coupled coincidence points, Alotaibi and Alsulami [68] proved the following result:

Theorem 6.6.5 ([68]). In addition to the hypotheses of Theorem 6.6.1, suppose that Assumption 3.3.1 holds. Then, F has unique coupled coincidence point.

Proof. By Theorem 6.6.1, there exists coupled coincidence points of F and g. Suppose that ($\varkappa$, y) and (ζ, t) are coupled coincidence points F and g, that is, $g\varkappa = F(\varkappa, y)$, $gy = F(y, \varkappa)$ and $g\zeta = F(\zeta, t)$, $gt = F(t, \zeta)$. To show that

$$g\varkappa = g\zeta \text{ and } gy = gt. \tag{6.6.2}$$

By assumption, there exists (u, v) in $X \times X$ that is comparable to ($\varkappa$, y) and (ζ, t). Define the sequences $\{gu_n\}$ and $\{gv_n\}$ as $u_0 = u$, $v_0 = v$ and $gu_{n+1} = F(u_n, v_n)$, $gv_{n+1} = F(v_n, u_n)$ for all n. Since (u, v) is comparable with ($\varkappa$, y), we assume that

$$(\varkappa, y) \succcurlyeq (u, v) = (u_0, v_0). \tag{6.6.3}$$

By mathematical induction, it is easy to obtain

$$(\varkappa, y) \succcurlyeq (u_n, v_n) \text{ for all n.} \tag{6.6.4}$$

By (6.6.1) and (6.6.4), we have

$$\varphi(d(g\varkappa, gu_{n+1})) = \varphi(d(F(\varkappa, y), F(u_n, v_n)))$$
$$\leq \frac{1}{2}\varphi(d(g\varkappa, gu_n) + d(gy, gv_n)) - \psi\left(\frac{d(g\varkappa, gu_n) + d(gy, gv_n)}{2}\right). \tag{6.6.5}$$

Similarly,

$$\varphi(d(gy, gv_{n+1})) = \varphi(d(F(y, \varkappa), F(v_n, u_n)))$$
$$\leq \frac{1}{2}\varphi(d(gy, gv_n) + d(g\varkappa, gu_n)) - \psi\left(\frac{d(gy, gv_n) + d(g\varkappa, gu_n)}{2}\right). \tag{6.6.6}$$

By (6.6.5), (6.6.6) and the property (φ_3) of φ, we get

$$\varphi(d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1}))$$
$$\leq \varphi(d(g\varkappa, gu_{n+1})) + \varphi(d(gy, gv_{n+1}))$$
$$\leq \varphi(d(g\varkappa, gu_n) + d(gy, gv_n)) - 2\psi\left(\frac{d(g\varkappa, gu_n) + d(gy, gv_n)}{2}\right). \tag{6.6.7}$$

Using the property of ψ, inequality (6.6.7) implies that

$$\varphi(d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1})) \leq \varphi(d(g\varkappa, gu_n) + d(gy, gv_n)),$$

which implies on using the monotone property of φ, that

$$d(g\varkappa, gu_{n+1}) + d(gy, gv_{n+1}) \leq d(g\varkappa, gu_n) + d(gy, gv_n),$$

so that $\{d(g\varkappa, gu_n) + d(gy, gv_n)\}$ is a decreasing sequence. Hence, there exists some $\alpha \geq 0$ such that $\lim_{n\to\infty}[d(g\varkappa, gu_n) + d(gy, gv_n)] = \alpha$. We claim that $\alpha = 0$. Suppose, on the contrary that $\alpha > 0$. Letting $n \to \infty$ in (6.6.7) and using the property of ψ, we get

$$\varphi(\alpha) \leq \varphi(\alpha) - 2 \lim_{n\to\infty}\psi\left(\frac{d(g\varkappa, gu_n) + d(gy, gv_n)}{2}\right) < \varphi(\alpha), \text{ a contradiction.}$$

Therefore, $\alpha = 0$, so that $\lim_{n\to\infty}[d(g\varkappa, gu_n) + d(gy, gv_n)] = 0$, which implies that

$$\lim_{n\to\infty} d(g\varkappa, gu_n) = 0 = \lim_{n\to\infty} d(gy, gv_n). \tag{6.6.8}$$

Similarly, we can get

$$\lim_{n\to\infty} d(g\zeta, gu_n) = 0 = \lim_{n\to\infty} d(gt, gv_n). \tag{6.6.9}$$

By (6.6.8) and (6.6.9), we obtain that $g\varkappa = g\zeta$ and $gy = gt$, that is, we proved (6.6.2).

Remark 6.6.4. (i) Note that the inequalities (6.6.5) and (6.6.6) do not follow by using (6.6.4) in (6.6.1), since (6.6.4) asserts that $(\varkappa, y) \succcurlyeq (u_n, v_n)$ for all n. But for (6.6.5) to hold, we require that $(g\varkappa, gy) \succcurlyeq (gu_n, gv_n)$. Similar is the case for the inequality (6.6.6).

(ii) The conclusion of Theorem 6.6.5 is that, Ғ and g have a unique coupled coincidence point. However, the proof only shows that $g\varkappa = g\zeta$ and $gy = gt$, where $(\varkappa, y)$ and (ζ, t) are assumed to be coupled coincidence points of Ғ and g. In order to reach the conclusion, it is necessary to show that $\varkappa = \zeta$ and $y = t$.

In view of Remark 6.6.4, we need to rectify Theorem 6.6.5. For this, we require the following results, stated again (for convenience):

Lemma 3.2.1. "The pair of compatible mappings $Ғ: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ commutes at their coincidence points".

Assumption 3.2.1 ([59]). "For every $(\varkappa, y), (\zeta, t) \in \mathfrak{X} \times \mathfrak{X}$, there exists a $(u, v) \in \mathfrak{X} \times \mathfrak{X}$ such that $(Ғ(u, v), Ғ(v, u))$ is comparable to $(Ғ(\varkappa, y), Ғ(y, \varkappa))$ and $(Ғ(\zeta, t), Ғ(t, \zeta))$".

We now rectify Theorem 6.6.5 as follows:

Theorem 6.6.6. In addition to the hypotheses of Theorem 6.6.1, suppose that Assumption 3.2.1 holds. Then, Ғ and g have a unique coupled coincidence point. Moreover, the mappings Ғ and g have a unique coupled fixed point.

Proof. By Theorem 6.6.1, there exist coupled coincidence points of Ғ and g. Let $(\varkappa, y)$ and (ζ, t) be coupled coincidence points of Ғ and g, so that $g\varkappa = Ғ(\varkappa, y)$, $gy = Ғ(y, \varkappa)$ and $g\zeta = Ғ(\zeta, t)$, $gt = Ғ(t, \zeta)$. We show that

$$g\varkappa = g\zeta \text{ and } gy = gt. \tag{6.6.10}$$

By Assumption 3.2.1, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Take $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$, $gv_1 = F(v_0, u_0)$. Then, inductively we can define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$ for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $t_0 = t$ and on the same way, define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$, $\{gt_n\}$ such that $gx_{n+1} = F(x_n, y_n)$, $gy_{n+1} = F(y_n, x_n)$ and $gz_{n+1} = F(z_n, t_n)$, $gt_{n+1} = F(t_n, z_n)$ for all $n \geq 0$.

Since $(F(u, v), F(v, u)) = (gu_1, gv_1)$ and $(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy)$ are comparable, then $gu_1 \succcurlyeq gx$ and $gv_1 \preccurlyeq gy$. Now, it is easy to obtain that (gu_n, gv_n) and (gx, gy) are comparable, so that $gu_n \succcurlyeq gx$ and $gv_n \preccurlyeq gy$ for all $n \geq 1$. By (6.6.1),

$$\varphi(d(gx, gu_{n+1})) = \varphi(d(F(x, y), F(u_n, v_n)))$$
$$\leq \frac{1}{2}\varphi(d(gx, gu_n) + d(gy, gv_n)) - \psi\left(\frac{d(gx, gu_n) + d(gy, gv_n)}{2}\right), \qquad (6.6.11)$$

which is inequality (6.6.5).

Similarly, we can obtain

$$\varphi(d(gy, gv_{n+1})) = \varphi(d(F(y, x), F(v_n, u_n)))$$
$$\leq \frac{1}{2}\varphi(d(gy, gv_n) + d(gx, gu_n)) - \psi\left(\frac{d(gy, gv_n) + d(gx, gu_n)}{2}\right), \qquad (6.6.12)$$

which is inequality (6.6.6).

Now, following the proof of Theorem 6.6.5, we can obtain that (6.6.10) holds.

Since (x, y) is a coupled coincidence point of the pair (F, g) of compatible mappings, by Lemma 3.2.1, it follows that

$$ggx = gF(x, y) = F(gx, gy) \text{ and } ggy = gF(y, x) = F(gy, gx). \qquad (6.6.13)$$

Denote $gx = r$, $gy = s$, then, by (6.6.13), we get

$$gr = F(r, s) \text{ and } gs = F(s, r). \qquad (6.6.14)$$

Thus, (r, s) is a coupled coincidence point of F and g. Then, by (6.6.10) with $z = r$ and $t = s$, it follows that

$$gr = r, \; gs = s. \qquad (6.6.15)$$

By (6.6.14) and (6.6.15), $r = gr = F(r, s)$ and $s = gs = F(s, r)$. Therefore, (r, s) is the coupled common fixed point of F and g. Hence, we obtained a coupled common fixed point of F and g. Also, if (e, f) is any coupled common fixed point of F and g, then, by (6.6.10), we have $e = ge = gr = r$ and $f = gf = gs = s$. This proves the uniqueness of coupled common fixed point of F and g.

Remark 6.6.5. Theorem 6.6.6 not only proves the uniqueness of coupled coincidence point of Ғ and g but also ensures the existence and uniqueness of coupled common fixed point of Ғ and g.

Similarly, on adding the Assumption 3.3.1 to the hypotheses of Theorem 6.6.3, Turkoglu and Sangurlu [169] asserts the existence and uniqueness of coupled fixed point for Ғ and g under the following result:

Theorem 6.6.7 ([169]). In addition to the hypotheses of Theorem 6.6.3, suppose that Assumption 3.3.1 holds, then, Ғ and g have a unique coupled fixed point.

Turkoglu and Sangurlu [169] had done the same mistakes in the formulation of Theorem 6.6.7 as done by Alotaibi and Alsulami [68] in Theorem 6.6.5. We note that these mistakes and errors can be rectified by redefining Theorem 6.6.7 as follows:

Theorem 6.6.8. In addition to the hypotheses of Theorem 6.6.4, suppose that Assumption 3.2.1 also holds. If the pair (Ғ, g) is compatible, then, Ғ and g have a unique coupled common fixed point.

Proof. Following the proof of Theorem 6.6.6, the result holds immediately.

6.7 AN ERROR IN A RECENT PAPER IN PGM-SPACES

In this section, we point out and rectify an error in a recent paper of Zhu et al. [120] in PGM-spaces.

Zhu et al. [120] called PGM-space as Menger PGM-space. The main result given by Zhu et al. [120] is as follows:

Theorem 6.7.1 (Zhu et al. [120]). "Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type and $\Delta \geq \Delta_p$, where Δ_p being the product norm. Let φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(\mathfrak{t}) < \infty$ for any ŧ > 0. Let Ғ: X × X → X and g: X → X be two mappings such that

$$G^*_{Ғ(\varkappa,y),\, Ғ(p,q),\, Ғ(h,l)}\big(\varphi(\mathfrak{t})\big) \geq \left[\Delta\big(G^*_{g\varkappa,\, gp,\, gh}(\mathfrak{t}), G^*_{gy,\, gq,\, gl}(\mathfrak{t})\big)\right]^{\frac{1}{2}}, \tag{6.7.1}$$

for all ϰ, y, p, q, h, l in X, where Ғ(X × X) ⊆ g(X), g is continuous and commutes with Ғ. Then, there exists a unique ɯ in X such that ɯ = gɯ = Ғ(ɯ, ɯ)".

Zhu et al. [120] gave the following example in support of Theorem 6.7.1:

Example 6.7.1 (Zhu et al. [120]). Suppose that $\Delta = \Delta_p$. Then Δ_p is a t-norm of H-type. Define a function G^*: X × X × X → $\mathbb{R}^+$ by $G^*_{\varkappa,y,z}(\mathfrak{t}) = \begin{cases} e^{-\frac{G(\varkappa,y,z)}{\mathfrak{t}}}, & \mathfrak{t} \geq 0, \\ 1, & \mathfrak{t} \leq 0, \end{cases}$ for all ϰ, y, z in X, where G(ϰ, y, z) = |ϰ − y| + |y − z| + |z − ϰ|.

Define the function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$ for $\mathfrak{t} \in \mathbb{R}^+$ and the mappings F: $X \times X \to X$ and g: $X \to X$ by

$$F(\varkappa, y) = \varkappa + y \quad \text{and} \quad g\varkappa = 4\varkappa \text{ for } \varkappa, y \text{ in } X.$$

Remark 6.7.1. Zhu et al. [120] claimed that Example 6.7.1 supports Theorem 6.7.1. In Theorem 6.7.1, the t-norm Δ is a t-norm of H-type with $\Delta \geq \Delta_p$ but in Example 6.7.1, the t-norm is Δ_p, which, in fact, is not a t-norm of H-type. Hence, Example 6.7.1 is incorrect.

We now construct an Example in support of Theorem 6.7.1 as follows:

Example 6.7.2. Let $X = \mathbb{R}^+$ and $\Delta = \Delta_m$, where Δ_m is the minimum t-norm which is a H-type t-norm with $\Delta \geq \Delta_p$. Define G^*: $X \times X \times X \to \Lambda^+$ by

$$G^*_{\varkappa,y,z}(\mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \leq 0, \\ e^{-\max\{|\varkappa-y|,|y-z|,|z-\varkappa|\}/\mathfrak{t}}, & \mathfrak{t} > 0, \end{cases}$$

for all $\varkappa$, y, z in X. Then, (X, G^*, Δ_m) is a complete Menger PGM-space.

Define F: $X \times X \to X$ and g: $X \to X$ by $F(\varkappa, y) = 1$ and $g\varkappa = \frac{2+\varkappa}{3}$ for all $\varkappa$, y in X. Clearly, g is continuous and $F(X \times X) \subseteq g(X)$. Also, the pair (F, g) is commutative, since for $\varkappa$, y in X, we have $gF(\varkappa, y) = 1 = F(g\varkappa, gy)$ and $gF(y, \varkappa) = 1 = F(gy, g\varkappa)$. Let φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be any gauge function with $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(\mathfrak{t}) < \infty$ for any $\mathfrak{t} > 0$.

Now, for $\varkappa$, y, z, p, q, l in X and $\mathfrak{t} > 0$, we verify that (6.7.1) holds, that is

$$G^*_{F(\varkappa,y),\, F(p,q),\, F(h,l)}(\varphi(\mathfrak{t})) \geq \left[\Delta\left(G^*_{g\varkappa,\, gp,\, gh}(\mathfrak{t}), G^*_{gy,\, gq,\, gl}(\mathfrak{t})\right)\right]^{\frac{1}{2}}.$$

For each $\varkappa$, y, z, p, q, l in X and $\mathfrak{t} > 0$, we have $G^*_{F(\varkappa,y),\, F(p,q),\, F(h,l)}(\varphi(\mathfrak{t})) = G^*_{1,1,1}(\varphi(\mathfrak{t})) = 1$, so that inequality (6.7.1) holds. Hence, all the conditions of Theorem 6.7.1 are satisfied. Now, by Theorem 6.7.1, g and F have a unique common fixed point in X, which is 1 in the present illustration.

Next, we give one more example in support of Theorem 6.7.1.

Example 6.7.3. Let $X = \mathbb{R}^+$ and $\Delta = \Delta_m$, where Δ_m is the minimum t-norm which is a H-type t-norm with $\Delta \geq \Delta_p$. Define H: $\mathbb{R}^+ \to \mathbb{R}^+$ and G^*: $X \times X \times X \to \Lambda^+$ respectively by

$$H(\mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} = 0, \\ 1, & \mathfrak{t} > 0, \end{cases} \quad \text{and} \quad G^*_{\varkappa,y,z}(\mathfrak{t}) = \begin{cases} H(\mathfrak{t}), & \varkappa = y = z, \\ \frac{\alpha\mathfrak{t}}{\alpha\mathfrak{t}+|\varkappa-y|+|y-z|+|z-\varkappa|}, & \text{otherwise}, \end{cases}$$

for all $\varkappa$, y, z in X with $\alpha > 0$. Then, (X, G^*, Δ_m) is a complete Menger PGM-space. Define the mappings Ꞙ: X × X → X and g: X → X by Ꞙ($\varkappa$, y) = β and g$\varkappa$ = $\frac{3\beta^2+\beta\varkappa}{2\beta+2\varkappa}$ for $\varkappa$, y in X and β is in ℝ is fixed. Also, the pair (Ꞙ, g) is commutative, g is continuous and Ꞙ(X × X) ⊆ g(X). Let φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be any gauge function with $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(f) < \infty$ for any f > 0.

Now, for $\varkappa$, y, z, p, q, l in X and f > 0, we verify that the inequality (6.7.1) holds.

For each $\varkappa$, y, z, p, q, l in X and f > 0, we have $G^*_{Ꞙ(\varkappa,y),\,Ꞙ(p,q),\,Ꞙ(h,l)}(\varphi(f))$ = $G^*_{\beta,\beta,\beta}(\varphi(f))$ = 1, so that the inequality (6.7.1) holds. Therefore, all the conditions of Theorem 6.7.1 are satisfied. Then, on applying Theorem 6.7.1, β is the unique common fixed point of Ꞙ and g.

6.8 SOME ERRORS IN A RECENT PAPER ON WEAKLY RELATED MAPPINGS

Recently, Singh and Jain [170] obtained coupled fixed points for non-decreasing mappings in POCMS using a partial order induced by some appropriate function ϕ. In this section, we point out and rectify some errors in [170].

Singh and Jain [170] gave the following notions:

Definition 6.8.1 ([170]). Let (X, ≼) be a poset and Ꞙ: X × X → X, g: X → X be two mappings. Then,

(i) Ꞙ is called **non-decreasing**, if

"for ($\varkappa_1$, y_1), ($\varkappa_2$, y_2) ∈ X × X and $\varkappa_1 \preccurlyeq \varkappa_2$, $y_1 \preccurlyeq y_2$ implies Ꞙ($\varkappa_1$, y_1) ≼ Ꞙ($\varkappa_2$, y_2)";

(ii) the pair (Ꞙ, g) called **weakly related**, if

" Ꞙ($\varkappa$, y) ≼ gꞘ($\varkappa$, y) and g$\varkappa$ ≼ Ꞙ(g$\varkappa$, gy), also Ꞙ(y, $\varkappa$) ≼ gꞘ(y, $\varkappa$) and gy ≼ Ꞙ(gy, g$\varkappa$) for all ($\varkappa$, y) ∈ X × X".

Lemma 6.8.1 ([170]). Let (X, ɖ) be a metric space and ϕ: X → ℝ a map. Define the relation "≼" on X as follows:

"$\varkappa \preccurlyeq$ y iff ɖ($\varkappa$, y) ≤ $\phi(\varkappa) - \phi$(y)".

Then "≼" is a partial order on X, called the **partial order induced by ϕ**.

Theorem 6.8.1 ([170]). "Let (X, ɖ) be a complete metric space, ϕ: X → ℝ be a bounded from above function and "≼" be the partial order induced by ϕ. Let Ꞙ: X × X → X be a non-decreasing continuous mapping on X such that there exist two elements $\varkappa_0$, y_0 in X with $\varkappa_0 \preccurlyeq$ Ꞙ($\varkappa_0$, y_0) and $y_0 \preccurlyeq$ Ꞙ(y_0, $\varkappa_0$). Then, Ꞙ has a coupled fixed point in X".

Theorem 6.8.2 ([170]). "Let (Ӽ, ɖ) be a complete metric space, ϕ: Ӽ → ℝ be a bounded from above function and "≼" be the partial order induced by ϕ. Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be two continuous mappings such that the pair (Ғ, g) is weakly related on Ӽ. If there exist two elements $\varkappa_0$, y_0 in Ӽ with $\varkappa_0 \preccurlyeq$ Ғ($\varkappa_0$, y_0) and $y_0 \preccurlyeq$ Ғ(y_0, $\varkappa_0$), then Ғ and g have a common coupled fixed point in Ӽ".

Singh and Jain [170] gave the following example in support of Theorem 6.8.1:

Example 6.8.1 ([170]). Let Ӽ = $\mathbb{R}^+$ and ɖ($\varkappa$, y) = $|\varkappa - y|$, then (Ӽ, d) is a complete metric space and "≼" is the usual ordering. Define ϕ: Ӽ → ℝ as $\phi(\varkappa)$ = 2$\varkappa$ and Ғ: Ӽ × Ӽ → Ӽ as Ғ($\varkappa$, y) = $\varkappa(1 + y)$. Take $\varkappa_0$ = 1 and y_0 = 0.

Remark 6.8.1. Singh and Jain [170] claimed that Example 6.8.1 supports Theorem 6.8.1. In Theorem 6.8.1, the function ϕ is assumed as bounded from above. But in Example 6.8.1, the function ϕ: Ӽ (= $\mathbb{R}^+$) → ℝ defined by $\phi(\varkappa)$ = 2$\varkappa$ for $\varkappa$ in Ӽ, is not bounded from above. Also, the order relation "≼" must be induced by ϕ. But in Example 6.8.1, it is considered to be the usual ordering.

Now, we rectify Example 6.8.1 as follows:

Example 6.8.2. Let Ӽ = [0, 1] and ɖ($\varkappa$, y) = $|\varkappa - y|$, then (Ӽ, ɖ) is a complete metric space. Define ϕ: Ӽ → ℝ by $\phi(\varkappa) = -2\varkappa$ for $\varkappa$ in Ӽ. Let the relation "≼" on Ӽ be defined as follows:

"$\varkappa \preccurlyeq y$ iff   ɖ($\varkappa$, y) ≤ $\phi(y) - \phi(\varkappa)$".

Then, "≼" is a partial order induced by ϕ. Clearly, ϕ is bounded from above on Ӽ.

Define Ғ: Ӽ × Ӽ → Ӽ by Ғ($\varkappa$, y) = $\frac{\varkappa(1+y)}{2}$ for all $\varkappa$, y in Ӽ. Then, Ғ is non-decreasing on Ӽ. Consider $\varkappa_0$ = 0 and y_0 = 1, then Ғ($\varkappa_0$, y_0) = $\frac{\varkappa_0(1+y_0)}{2}$ = 0 and Ғ(y_0, $\varkappa_0$) = $\frac{y_0(1+\varkappa_0)}{2} = \frac{1}{2}$. Finally, we claim that $\varkappa_0 \preccurlyeq$ Ғ($\varkappa_0$, y_0) and $y_0 \preccurlyeq$ Ғ(y_0, $\varkappa_0$).

Now, $\varkappa_0 \preccurlyeq$ Ғ($\varkappa_0$, y_0) iff ɖ($\varkappa_0$, Ғ($\varkappa_0$, y_0)) ≤ ϕ(Ғ($\varkappa_0$, y_0)) – $\phi(\varkappa_0)$

iff   ɖ(0, 0) = 0 ≤ $\phi(0) - \phi(0)$ = 0, which is true.

Also, $y_0 \preccurlyeq$ Ғ(y_0, $\varkappa_0$) iff ɖ(y_0, Ғ(y_0, $\varkappa_0$)) ≤ ϕ(Ғ(y_0, $\varkappa_0$)) – $\phi(y_0)$

iff   ɖ$\left(1, \frac{1}{2}\right) \leq \phi\left(\frac{1}{2}\right) - \phi(1)$

iff $\frac{1}{2} \leq (-2)\left(\frac{1}{2}\right) - (-2)(1) = 1$, which is again true.

Therefore, all the conditions of Theorem 6.8.1 are satisfied. Now, by Theorem 6.8.1, (0, 0) is a coupled fixed point of Ғ.

Remark 6.8.2. In Example 6.8.2, since $1 \preccurlyeq \frac{1}{2}$ but $1 \nleq \frac{1}{2}$, so the partial order "$\preccurlyeq$" induced by ϕ is not the usual ordering "$\leq$".

Singh and Jain [170] supported Theorem 6.8.2 by using the following example:

Example 6.8.3 ([170]). Let $X = \mathbb{R}^+$ and $d(x, y) = |x - y|$, then (X, d) is a complete metric space and "$\preccurlyeq$" is the usual ordering. Define ϕ: $X \to \mathbb{R}$ as $\phi(x) = 2x$ and F: $X \times X \to X$ and g: $X \to X$ as $F(x, y) = x + |\sin(xy)|$ and $gx = 5x$. Take $x_0 = 1$ and $y_0 = 0$, then $F(x_0, y_0) = 1$ and $F(y_0, x_0) = 0$, so that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \preccurlyeq F(y_0, x_0)$. Also, the pair (F, g) is weakly related.

Remark 6.8.3. Singh and Jain [170] claimed that Example 6.8.3 supports Theorem 6.8.2. In Theorem 6.8.2, the function ϕ is assumed as bounded from above. But in Example 6.8.3, the function ϕ: X $(= \mathbb{R}^+) \to \mathbb{R}$ defined by $\phi(x) = 2x$ for x in X, is not bounded from above. Also, the order relation "$\preccurlyeq$" must be induced by ϕ. But in Example 6.8.3, it is considered to be the usual ordering.

Now, we rectify Example 6.8.3 as follows:

Example 6.8.4. Let $X = [0, 1]$ and $d(x, y) = |x - y|$, then (X, d) is a complete metric space. Let ϕ: $X \to \mathbb{R}$ be the mapping defined by $\phi(x) = -2x$ for x in X. Let the relation "$\preccurlyeq$" on X be defined as follows:

$$\text{"}x \preccurlyeq y \quad \text{iff} \quad d(x, y) \leq \phi(y) - \phi(x)\text{"}.$$

Then, "$\preccurlyeq$" is a partial order induced by ϕ. Clearly, ϕ is bounded from above on X. Also, here $1 \preccurlyeq \frac{1}{2}$, so "$\preccurlyeq$" is not the usual order "$\leq$".

Define F: $X \times X \to X$ and g: $X \to X$ by $F(x, y) = \frac{x(1+y)}{4}$ and $gx = \frac{x}{2}$ for x, y in X. Now, $gF(x, y) = \frac{x(1+y)}{8}$, $F(gx, gy) = F\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{x(2+y)}{16}$, $gF(y, x) = \frac{y(1+x)}{8}$, $F(gy, gx) = F\left(\frac{y}{2}, \frac{x}{2}\right) = \frac{y(2+x)}{16}$ for x, y in X.

We now show that the pair (F, g) is weakly related.

For, we consider the following:

i) $F(x, y) \preccurlyeq gF(x, y)$ iff $d(F(x, y), gF(x, y)) \leq \phi(gF(x, y)) - \phi(F(x, y))$

$$\text{iff} \quad \left|\frac{x(1+y)}{4} - \frac{x(1+y)}{8}\right| \leq \phi\left(\frac{x(1+y)}{8}\right) - \phi\left(\frac{x(1+y)}{4}\right)$$

$$\text{iff} \quad \frac{x(1+y)}{8} \leq -\frac{x(1+y)}{4} + \frac{x(1+y)}{2}$$

$$\text{iff} \quad \frac{x(1+y)}{8} \leq \frac{x(1+y)}{4},$$

which is true for x, y in X.

ii) $g\varkappa \preccurlyeq Ғ(g\varkappa, gy)$ iff $d(g\varkappa, Ғ(g\varkappa, gy)) \leq \phi(Ғ(g\varkappa, gy)) - \phi(g\varkappa)$

iff $\left|\frac{\varkappa}{2} - \frac{\varkappa(2+y)}{16}\right| \leq \phi\left(\frac{\varkappa(2+y)}{16}\right) - \phi\left(\frac{\varkappa}{2}\right)$

iff $\left|\frac{6\varkappa - \varkappa y}{16}\right| \leq -\frac{\varkappa(2+y)}{8} + x$

iff $\left|\frac{6\varkappa - \varkappa y}{16}\right| \leq \frac{6\varkappa - \varkappa y}{8}$, which is true for $\varkappa$, y in Ꭓ.

Similarly, we can get $Ғ(y, \varkappa) \preccurlyeq gҒ(y, \varkappa)$ and $gy \preccurlyeq Ғ(gy, g\varkappa)$ for all $\varkappa$, y in Ꭓ.

Therefore, the pair (Ғ, g) is weakly related.

Let $\varkappa_0 = 0$, $y_0 = 1$, then $Ғ(\varkappa_0, y_0) = 0$ and $Ғ(y_0, \varkappa_0) = \frac{1}{4}$.

Finally, we verify that $\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ and $y_0 \preccurlyeq Ғ(y_0, \varkappa_0)$.

For, $\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ iff $d(\varkappa_0, Ғ(\varkappa_0, y_0)) \leq \phi(Ғ(\varkappa_0, y_0)) - \phi(\varkappa_0)$

iff $d(0, 0) \leq \phi(0) - \phi(0)$, which is true.

Also $y_0 \preccurlyeq Ғ(y_0, \varkappa_0)$ iff $d(y_0, Ғ(y_0, \varkappa_0)) \leq \phi(Ғ(y_0, \varkappa_0)) - \phi(y_0)$

iff $d\left(1, \frac{1}{4}\right) \leq \phi\left(\frac{1}{4}\right) - \phi(1)$

iff $\frac{3}{4} \leq (-2)\left(\frac{1}{4}\right) - (-2)(1)$

iff $\frac{3}{4} \leq -\frac{1}{2} + 2 = \frac{3}{2}$, which is true.

Therefore, all the conditions of Theorem 6.8.2 are satisfied. By Theorem 6.8.2, (0, 0) is the common coupled fixed point of the pair (Ғ, g).

FRAMEWORK OF CHAPTER - VII

In this chapter, we prove some fixed point results and obtain some corresponding coupled fixed point results in POMS. Further, some results for mappings lacking MgMP are also obtained.

PUBLISHED WORK:

(1) TWMS Journal of Applied and Engineering Mathematics, 7(1) (2017), pp. 110-130.

CONFERENCE:

(1) One paper presented in the International Conference ICAA-2015 held during Dec 19-21, 2015 at AMU, Aligarh (U.P.).

CHAPTER – VII

FIXED POINT AND COUPLED FIXED POINT RESULTS

In this chapter, we prove some fixed point and coupled fixed point results in POMS. The results obtained are generalizations of a number of existing works. This chapter consists of four sections. Section 7.1 presents some already existing contractions in POMS. In section 7.2, we prove some fixed point results for generalized weak $(\psi > \phi)$ – contraction mappings. Section 7.3 consists of the application of the results established in section 7.2 to coupled fixed point results. In section 7.4, we establish some coupled coincidence point and coupled common fixed point results for the pair of mappings lacking MgMP.

Author's Original Contributions In This Chapter Are:

Theorems: 7.2.1, 7.2.2, 7.2.3, 7.2.4, 7.2.5, 7.3.1, 7.3.2, 7.4.1, 7.4.3.

Definitions: 7.2.1, 7.2.2.

Corollaries: 7.2.1, 7.2.3, 7.4.1.

Examples: 7.2.1, 7.2.2, 7.4.1, 7.4.2.

Remarks: 7.2.1, 7.2.2, 7.3.1, 7.3.2, 7.4.1, 7.4.2, 7.4.3, 7.4.4, 7.4.5.

7.1. SOME RECENT CONTRACTIONS

In this section, we mention some important contractions that have been used by different authors to obtain results in fixed point theory and coupled fixed point theory.

For the sake of convenience, we also cite the serials of theorems and the relevant contractions used in the previous chapters of the present work.

Let (Ϫ, ≼, ɖ) be a POMS and ϰ, y, u, v ∈ Ϫ. Let ħ, g be the self mappings on Ϫ.

(i) Ran and Reurings [40] (Theorem 2.1.3, contraction (2.1.3));

Nieto and L$\acute{o}$pez [41] (Theorem 2.1.4, contraction (2.1.6)):

$$ɖ(ħϰ, ħy) \leq ķɖ(ϰ, y), \text{ for } ϰ \succcurlyeq y, \text{ where } 0 < ķ < 1; \quad (7.1.1)$$

(ii) Harjani and Sadarangani [47] (Theorem 2.1.8, contraction (2.1.9)):

$$ɖ(ħϰ, ħy) \leq ɖ(ϰ, y) - \psi(ɖ(ϰ, y)), \quad (7.1.2)$$

for $ϰ \succcurlyeq y$, where ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function such that ψ is positive in $\mathbb{R}^+\backslash\{0\}$, $\psi(0) = 0$ and $\lim_{ƫ\to\infty}\psi(ƫ) = \infty$;

(iii) Harjani and Sadarangani [48] (Theorem 2.1.9, contraction (2.1.10)):

$$\psi(ɖ(ħϰ, ħy)) \leq \psi(ɖ(ϰ, y)) - \phi(ɖ(ϰ, y)), \quad (7.1.3)$$

for $x \succcurlyeq y$, where ψ, ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ are ADF;

(iv) Amini-Harandi and Emami [53] (Theorem 2.1.13, contraction (2.1.2)):

$$d(\hbar\varkappa, \hbar y) \leq \beta(d(\varkappa, y))\, d(\varkappa, y), \tag{7.1.4}$$

for $\varkappa \succcurlyeq y$, where $\beta \in \mathfrak{R} = \{\beta \mid \beta: \mathbb{R}^+ \to [0, 1), \beta(t_n) \to 1 \text{ implies } t_n \to 0\}$;

(v) Ćirić et al. [52] (Theorem 2.1.12, contraction (2.1.13)):

$$d(\hbar\varkappa, \hbar y) \leq \max\left\{\begin{matrix}\varphi\big(d(g\varkappa, gy)\big), \varphi\big(d(g\varkappa, \hbar\varkappa)\big), \varphi\big(d(gy,\hbar y)\big) \\ \varphi\left(\frac{d(g\varkappa, \hbar y)+d(gy, \hbar\varkappa)}{2}\right)\end{matrix}\right\}, \tag{7.1.5}$$

for $g\varkappa \succcurlyeq gy$, where φ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\varphi(t) < t$ for each $t > 0$.

In the context of coupled fixed point theory, for the mappings F: $X \times X \to X$ and g: $X \to X$, the following contractions have been enjoyed by various authors:

(vi) Bhaskar and Lakshmikantham [55] (Theorem 2.1.14, contraction (2.1.14)):

$$d(F(\varkappa, y), F(u, v)) \leq \frac{k}{2}\,[d(\varkappa, u) + d(y, v)], \text{ where } k \in [0, 1); \tag{7.1.6}$$

for all $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$.

(vii) Harjani et al. [58] (Theorem 2.1.15, contraction (2.1.15)):

$$\varphi(d(F(\varkappa, y), F(u, v))) \leq \varphi(\max\{d(\varkappa, u), d(y, v)\}) - \phi(\max\{d(\varkappa, u), d(y, v)\}), \tag{7.1.7}$$

for all $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, where φ, ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ are ADF;

(viii) Berinde [149] (Theorem 3.1.1, contraction (3.1.1)):

$$d(F(\varkappa, y), F(u, v)) + d(F(y, \varkappa), F(v, u)) \leq k\,[d(\varkappa, u) + d(y, v)], \tag{7.1.8}$$

for all $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, where $k \in [0, 1)$;

(ix) Rasouli and Bahrampour [70] (Theorem 2.1.23, contraction (2.1.22)):

$$d(F(\varkappa, y), F(u, v)) \leq \beta(\max\{d(\varkappa, u), d(y, v)\}) \max\{d(\varkappa, u), d(y, v)\}, \tag{7.1.9}$$

for all $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, where $\beta \in \mathfrak{R}$;

(x) Choudhury et al. [56] (Theorem 2.1.18, contraction (2.1.17)):

$$\psi(d(F(\varkappa, y), F(u, v))) \leq \psi(\max\{d(g\varkappa, gu), d(gy, gv)\}) - \phi(\max\{d(g\varkappa, gu), d(gy, gv)\}), \tag{7.1.10}$$

for $g\varkappa \succcurlyeq gu$, $gy \preccurlyeq gv$, where ψ, ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ be such that ψ is an ADF and ϕ is continuous and $\phi(t) = 0$ iff $t = 0$;

(xi) Jain et al. [159] (Corollary 3.2.1, contraction (3.2.23)):

$$d(F(\varkappa, y), F(u, v)) + d(F(y, \varkappa), F(v, u)) \leq k\,[d(g\varkappa, gu) + d(gy, gv)], \tag{7.1.11}$$

for $g\varkappa \succcurlyeq gu$, $gy \preccurlyeq gv$, where $k \in [0, 1)$;

(xii) Luong and Thuan [69] (Theorem 2.1.21, contraction (2.1.20)):

$$ꝱ(Ғ(\varkappa, y), Ғ(u, v)) \leq \alpha\, ꝱ(\varkappa, u) + \beta\, ꝱ(y, v) + Ł \min\begin{Bmatrix} ꝱ(Ғ(\varkappa, y), u), ꝱ(Ғ(u, v), \varkappa), \\ ꝱ(Ғ(\varkappa, y), \varkappa), ꝱ(Ғ(u, v), u) \end{Bmatrix}, \tag{7.1.12}$$

for $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, where $\alpha, \beta, Ł \geq 0$ with $\alpha + \beta < 1$;

(xiii) Karapinar et al. [57] (Theorem 2.1.22, contraction (2.1.21)):

$$ꝱ(Ғ(\varkappa, y), Ғ(u, v)) \leq \phi(\max\{ꝱ(g\varkappa, gu), ꝱ(gy, gv)\}) + Ł \min\begin{Bmatrix} ꝱ(Ғ(\varkappa, y), gu), ꝱ(Ғ(u, v), g\varkappa), \\ ꝱ(Ғ(\varkappa, y), gx), ꝱ(Ғ(u, v), gu) \end{Bmatrix}, \tag{7.1.13}$$

for $g\varkappa \succcurlyeq gu$, $gy \preccurlyeq gv$, where $Ł \geq 0$ and $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with the condition that $\phi(f) < f$ for all $f > 0$.

Recently, Haghi et al. [165] proved a lemma given below, which is useful for us in developing our results:

Lemma 7.1.1 ([165]). Let Ӽ be a nonempty set and ħ: Ӽ → Ӽ a function. Then there exists a subset A ⊆ Ӽ such that ħ(A) = ħ(Ӽ) and ħ: A → Ӽ is one-to-one.

7.2. GENERALIZED WEAK ($\psi > \phi$) – CONTRACTIONS

In this section, we prove fixed point results for generalized weak ($\psi > \phi$) – contraction mappings in the setup of POMS. The results obtained are the generalizations of the works of Ran and Reurings [40], Nieto and L$\acute{o}$pez [41], Harjani and Sadarangani [47], Harjani and Sadarangani [48], Amini-Harandi and Emami [53] and $\acute{C}$iri$\acute{c}$ et al. [52].

We first introduce the notion of generalized weak ($\psi > \phi$) – contraction as follows:

Definition 7.2.1. Let (Ӽ, ꝱ) be a metric space. A self mapping ħ on Ӽ is called a **generalized weak ($\psi > \phi$) – contraction** if it satisfies the following condition:

$$\psi\big(ꝱ(ħ\varkappa, ħy)\big) \leq M(\varkappa, y), \tag{7.2.1}$$

for $\varkappa$, y ∈ Ӽ, where

$$M(\varkappa, y) = \max\left\{\phi\big(ꝱ(\varkappa, y)\big), \phi\big(ꝱ(\varkappa, ħx)\big), \phi\big(ꝱ(y, ħy)\big), \phi\left(\frac{ꝱ(\varkappa, ħy) + ꝱ(y, ħ\varkappa)}{2}\right)\right\}, \tag{7.2.2}$$

with ψ being an ADF and $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a continuous function such that $\psi(f) > \phi(f)$ for all $f > 0$.

Lemma 7.2.1. ([171]). If ψ is an ADF and $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with the condition $\psi(f) > \phi(f)$ for all $f > 0$, then $\phi(0) = 0$.

We now give our results as follows:

Theorem 7.2.1. Let (Ӽ, ≼, ḑ) be a POCMS. Let ħ be a non-decreasing and continuous self mapping on Ӽ such that ħ is a generalized weak ($\psi > \phi$) – contraction mapping for all $\varkappa$, y ∈ Ӽ with $\varkappa \succcurlyeq y$. If there exists $\varkappa_0 \in$ Ӽ such that $\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then ħ has a fixed point.

Proof. Since ħ is non-decreasing, by induction, for n ≥ 0, we get

$$\hbar^n\varkappa_0 \preccurlyeq \hbar^{n+1}\varkappa_0.$$

Set $\varkappa_{n+1} = \hbar\varkappa_n = \hbar^n\varkappa_0$, so that we have

$$\varkappa_n \preccurlyeq \varkappa_{n+1}. \tag{7.2.3}$$

W.L.O.G., assume $\varkappa_n \neq \hbar\varkappa_n$ for all n ∈ ℕ, otherwise $\varkappa_n$ is a fixed point of the mapping ħ for some n ∈ ℕ. We will first show that

$$ḑ(\varkappa_{n+1}, \varkappa_n) < ḑ(\varkappa_{n-1}, \varkappa_n), \text{ for all } n \in \mathbb{N}. \tag{7.2.4}$$

For n ∈ ℕ, since $\varkappa_n$ and $\varkappa_{n+1}$ are comparable, by given hypothesis, we get

$$\psi\big(ḑ(\varkappa_{n+1}, \varkappa_n)\big) = \psi\big(ḑ(\hbar\varkappa_n, \hbar\varkappa_{n-1})\big)$$

$$\leq \max\left\{\begin{matrix}\phi\big(ḑ(\varkappa_n, \varkappa_{n-1})\big), \phi\big(ḑ(\varkappa_n, \hbar\varkappa_n)\big), \phi\big(ḑ(\varkappa_{n-1}, \hbar\varkappa_{n-1})\big), \\ \phi\left(\frac{ḑ(\varkappa_n, \hbar\varkappa_{n-1}) + ḑ(\varkappa_{n-1}, \hbar\varkappa_n)}{2}\right)\end{matrix}\right\} = M_n, \tag{7.2.5}$$

where $M_n = \max\left\{\phi\big(ḑ(\varkappa_n, \varkappa_{n-1})\big), \phi\big(ḑ(\varkappa_n, \varkappa_{n+1})\big), \phi\left(\frac{ḑ(\varkappa_{n-1}, \varkappa_{n+1})}{2}\right)\right\}$. (7.2.6)

Case 1. If $M_n = \phi\big(ḑ(\varkappa_n, \varkappa_{n-1})\big)$,

then $\psi\big(ḑ(\varkappa_{n+1}, \varkappa_n)\big) \leq \phi\big(ḑ(\varkappa_n, \varkappa_{n-1})\big) < \psi\big(ḑ(\varkappa_n, \varkappa_{n-1})\big)$. Using the monotone property of ψ, we have $ḑ(\varkappa_{n+1}, \varkappa_n) < ḑ(\varkappa_n, \varkappa_{n-1})$.

Case 2. If $M_n = \phi\big(ḑ(\varkappa_n, \varkappa_{n+1})\big)$,

then $\psi\big(ḑ(\varkappa_{n+1}, \varkappa_n)\big) \leq \phi\big(ḑ(\varkappa_n, \varkappa_{n+1})\big) < \psi\big(ḑ(\varkappa_n, \varkappa_{n+1})\big)$, a contradiction.

Case 3. If $M_n = \phi\left(\frac{ḑ(\varkappa_{n-1}, \varkappa_{n+1})}{2}\right)$,

then $\psi\big(ḑ(\varkappa_{n+1}, \varkappa_n)\big) \leq \phi\left(\frac{ḑ(\varkappa_{n-1}, \varkappa_{n+1})}{2}\right) < \psi\left(\frac{ḑ(\varkappa_{n-1}, \varkappa_{n+1})}{2}\right)$. Now, using the monotone property of ψ, it follows that

$$ḑ(\varkappa_{n+1}, \varkappa_n) < \frac{ḑ(\varkappa_{n-1}, \varkappa_{n+1})}{2} \leq \frac{1}{2}\big(ḑ(\varkappa_{n-1}, \varkappa_n) + ḑ(\varkappa_n, \varkappa_{n+1})\big),$$

which implies $ḑ(\varkappa_{n+1}, \varkappa_n) < ḑ(\varkappa_{n-1}, \varkappa_n)$. Hence, (7.2.4) holds, so that $\{Ɽ_n\}$ is a decreasing sequence, where $Ɽ_n = ḑ(\varkappa_n, \varkappa_{n+1})$. Consequently, there exists some Ɽ ≥ 0 such that

$$\lim_{n\to\infty} Ɽ_n = Ɽ. \tag{7.2.7}$$

We claim that $\mathfrak{R} = 0$.

Using triangle inequality, we have

$$\frac{1}{2}d(\varkappa_{n-1}, \varkappa_{n+1}) \leq \frac{1}{2}\left(d(\varkappa_{n-1}, \varkappa_n) + d(\varkappa_n, \varkappa_{n+1})\right).$$

Hence, by (7.2.4), we obtain that

$$\frac{1}{2}d(\varkappa_{n-1}, \varkappa_{n+1}) < d(\varkappa_{n-1}, \varkappa_n). \quad (7.2.8)$$

Letting the upper limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{1}{2}d(\varkappa_{n-1}, \varkappa_{n+1}) \leq \lim_{n\to\infty} d(\varkappa_{n-1}, \varkappa_n). \quad (7.2.9)$$

On setting

$$\limsup_{n\to\infty} \frac{1}{2}d(\varkappa_{n-1}, \varkappa_{n+1}) = \mathfrak{b}, \quad (7.2.10)$$

we can obtain that $0 \leq \mathfrak{b} \leq \mathfrak{R}$. Now, taking the upper limit in (7.2.5) and using continuity of ψ and ϕ, we get

$$\psi(\lim_{n\to\infty} d(\varkappa_{n+1}, \varkappa_n)) \leq \max\left\{\begin{matrix}\phi(\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n-1})), \phi(\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n+1})), \\ \phi(\limsup_{n\to\infty} \frac{1}{2}d(\varkappa_{n-1}, \varkappa_{n+1}))\end{matrix}\right\}. \quad (7.2.11)$$

Now, using (7.2.7) and (7.2.10) in (7.2.11), we get

$$\psi(\mathfrak{R}) \leq \max\{\phi(\mathfrak{R}), \phi(\mathfrak{R}), \phi(\mathfrak{b})\} = \mathfrak{m} \text{ (say)}. \quad (7.2.12)$$

If $\mathfrak{m} = \phi(\mathfrak{b})$, then $\psi(\mathfrak{R}) \leq \phi(\mathfrak{b})$. If $\mathfrak{b} = 0$, we have $\psi(\mathfrak{R}) \leq \phi(0)$ implying that $\psi(\mathfrak{R}) = 0$ and hence $\mathfrak{R} = 0$, otherwise for $\mathfrak{b} > 0$, we have $\psi(\mathfrak{R}) < \phi(\mathfrak{b}) < \psi(\mathfrak{b})$ implying that $\mathfrak{R} < \mathfrak{b}$, a contradiction. If we suppose that $\mathfrak{R} > 0$, then we have $\psi(\mathfrak{R}) \leq \phi(\mathfrak{R}) < \psi(\mathfrak{R})$, a contradiction. Therefore, $\mathfrak{R} = 0$, so that

$$\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n+1}) = \lim_{n\to\infty} \mathfrak{R}_n = 0. \quad (7.2.13)$$

Next, we claim that $\{\varkappa_n\}$ is a Cauchy sequence. On the contrary, suppose $\{\varkappa_n\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ and sequences of integers $\{l(k)\}$, $\{m(k)\}$, such that

$$m(k) > l(k) \geq k$$

with $\mathfrak{r}_k = d\left(\varkappa_{l(k)}, \varkappa_{m(k)}\right) \geq \varepsilon$ for $k \in \mathbb{N}$. (7.2.14)

We may further assume that

$$d\left(\varkappa_{l(k)}, \varkappa_{m(k)-1}\right) < \varepsilon, \quad (7.2.15)$$

by choosing $m(k)$ to be the smallest number exceeding $l(k)$ for which (7.2.14) holds. By (7.2.14), (7.2.15) and using triangle inequality, we have

$$\varepsilon \leq \mathfrak{r}_k \leq d\left(\varkappa_{l(k)}, \varkappa_{m(k)-1}\right) + d\left(\varkappa_{m(k)-1}, \varkappa_{m(k)}\right) < \varepsilon + d\left(\varkappa_{m(k)-1}, \varkappa_{m(k)}\right). \quad (7.2.16)$$

Taking $k \to \infty$ in (7.2.16) and using (7.2.13), we have

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{7.2.17}$$

Since

$$r_k = d(\varkappa_{l(k)}, \varkappa_{m(k)}) \le d(\varkappa_{l(k)}, \varkappa_{l(k)+1}) + d(\varkappa_{l(k)+1}, \varkappa_{m(k)+1}) + d(\varkappa_{m(k)+1}, \varkappa_{m(k)})$$
$$= R_{l(k)} + R_{m(k)} + d(\varkappa_{l(k)+1}, \varkappa_{m(k)+1}),$$

then, by the monotone property of ψ, we obtain

$$\psi(r_k) \le \psi\left(R_{l(k)} + R_{m(k)} + d(\hbar\varkappa_{l(k)}, \hbar\varkappa_{m(k)})\right),$$

on letting $k \to \infty$ and using the continuity of ψ, (7.2.13) and (7.2.17), we have

$$\psi(\varepsilon) \le \psi\left(\lim_{k\to\infty} d(\hbar\varkappa_{l(k)}, \hbar\varkappa_{m(k)})\right) = \lim_{k\to\infty} \psi\left(d(\hbar\varkappa_{l(k)}, \hbar\varkappa_{m(k)})\right). \tag{7.2.18}$$

Also, it can be easily obtained that

$$\lim_{k\to\infty} \frac{d(\varkappa_{l(k)}, \varkappa_{m(k)+1}) + d(\varkappa_{m(k)}, \varkappa_{l(k)+1})}{2} = \varepsilon. \tag{7.2.19}$$

Since m(k) > l(k), so $\varkappa_{m(k)}$ and $\varkappa_{l(k)}$ are comparable, then by given hypothesis, we can get

$$\psi\left(d(\varkappa_{l(k)+1}, \varkappa_{m(k)+1})\right)$$
$$= \psi\left(d(\hbar\varkappa_{l(k)}, \hbar\varkappa_{m(k)})\right)$$
$$\le \max\left\{\begin{matrix}\phi\left(d(\varkappa_{l(k)}, \varkappa_{m(k)})\right), \phi\left(d(\varkappa_{l(k)}, \hbar\varkappa_{l(k)})\right), \phi\left(d(\varkappa_{m(k)}, \hbar\varkappa_{m(k)})\right), \\ \phi\left(\frac{d(\varkappa_{l(k)}, \hbar\varkappa_{m(k)}) + d(\varkappa_{m(k)}, \hbar\varkappa_{l(k)})}{2}\right)\end{matrix}\right\}$$
$$= \max\left\{\begin{matrix}\phi\left(d(\varkappa_{l(k)}, \varkappa_{m(k)})\right), \phi\left(d(\varkappa_{l(k)}, \varkappa_{l(k)+1})\right), \\ \phi\left(d(\varkappa_{m(k)}, \varkappa_{m(k)+1})\right), \phi\left(\frac{d(\varkappa_{l(k)}, \varkappa_{m(k)+1}) + d(\varkappa_{m(k)}, \varkappa_{l(k)+1})}{2}\right)\end{matrix}\right\} \tag{7.2.20}$$
$$= \max\left\{\phi(r_k), \phi(R_{l(k)}), \phi(R_{m(k)}), \left(\frac{d(\varkappa_{l(k)}, \varkappa_{m(k)+1}) + d(\varkappa_{m(k)}, \varkappa_{l(k)+1})}{2}\right)\right\}. \tag{7.2.21}$$

Taking $k \to \infty$ in (7.2.21) and using (7.2.18), we get

$$\psi(\varepsilon) \le \lim_{k\to\infty} \psi\left(d(\hbar\varkappa_{l(k)}, \hbar\varkappa_{m(k)})\right)$$
$$\le \lim_{k\to\infty} \max\left\{\phi(r_k), \phi(R_{l(k)}), \phi(R_{m(k)}), \phi\left(\frac{d(\varkappa_{l(k)}, \varkappa_{m(k)+1}) + d(\varkappa_{m(k)}, \varkappa_{l(k)+1})}{2}\right)\right\},$$

then, using the continuity of ϕ and (7.2.13), (7.2.17), (7.2.19), we get

$$\psi(\varepsilon) \le \max\{\phi(\varepsilon), \phi(0), \phi(0), \phi(\varepsilon)\} = \phi(\varepsilon) < \psi(\varepsilon),$$

a contradiction. Therefore, $\{\varkappa_n\}$ is a Cauchy sequence. By the completeness of X, there exists some $u \in X$ such that

$$\lim_{n\to\infty} \varkappa_n = u. \tag{7.2.22}$$

Finally, we show that $ɯ$ is a fixed point of $ħ$.

By continuity of $ħ$, we have

$$ɯ = \lim_{n\to\infty} \varkappa_{n+1} = \lim_{n\to\infty} ħ\varkappa_n = ħɯ.$$

This completes the proof of our result.

Now, by assuming the Assumption 2.1.2 in $Ҳ$, we shall now prove that Theorem 7.2.1 still holds for a non-continuous function $ħ$. For the sake of convenience, we again give Assumption 2.1.2 as follows:

Assumption 2.1.2 ([41]). $Ҳ$ has the property that: "if a non-decreasing sequence $\{\varkappa_n\} \subseteq Ҳ$ converges to $\varkappa$, then $\varkappa_n \preccurlyeq \varkappa$ for all n".

Now, we give our result:

Theorem 7.2.2. Let $(Ҳ, \preccurlyeq, ȡ)$ be a POCMS. Assume that the Assumption 2.1.2 holds in $Ҳ$. Let $ħ$ be a non-decreasing self mapping on $Ҳ$ such that $ħ$ is a generalized weak $(\psi > \phi)$ – contraction mapping for all $\varkappa$, y $\in Ҳ$ with $\varkappa \succcurlyeq$ y. If there exists some $\varkappa_0 \in Ҳ$ such that $\varkappa_0 \preccurlyeq ħ\varkappa_0$, then $ħ$ has a fixed point in $Ҳ$.

Proof. Following the proof of Theorem 7.2.1, we have to only verify that $ħɯ = ɯ$. On the contrary, suppose that $ɯ \neq ħɯ$. Since $\varkappa_n \preccurlyeq \varkappa_{n+1}$ for all $n \in \mathbb{N}$ and $\varkappa_n \to u$ as $n \to \infty$, so by given hypotheses $\varkappa_n \preccurlyeq ɯ$ for all $n \in \mathbb{N}$.

Now, by hypothesis, we have

$$\begin{aligned}
&\psi\big(ȡ(ħɯ, \varkappa_{n+1})\big)\\
&\quad = \psi\big(ȡ(ħɯ, ħ\varkappa_n)\big)\\
&\quad \leq \max\Big\{\phi\big(ȡ(ɯ, \varkappa_n)\big), \phi\big(ȡ(ɯ, ħɯ)\big), \phi\big(ȡ(\varkappa_n, ħ\varkappa_n)\big), \phi\Big(\tfrac{ȡ(ɯ, ħ\varkappa_n)+ȡ(\varkappa_n, ħɯ)}{2}\Big)\Big\}\\
&\quad = \max\Big\{\phi\big(ȡ(ɯ, \varkappa_n)\big), \phi\big(ȡ(ɯ, ħɯ)\big), \phi\big(ȡ(\varkappa_n, \varkappa_{n+1})\big), \phi\Big(\tfrac{ȡ(ɯ, \varkappa_{n+1})+ȡ(\varkappa_n, ħɯ)}{2}\Big)\Big\}, \qquad (7.2.23)
\end{aligned}$$

then, letting $n \to \infty$ in (7.2.23) and using the continuity of ψ, ϕ we obtain

$$\begin{aligned}
\psi\big(ȡ(ħɯ, ɯ)\big) &\leq \max\Big\{\phi(0), \phi\big(ȡ(ɯ, ħɯ)\big), \phi(0), \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big)\Big\}\\
&= \max\Big\{\phi\big(ȡ(ɯ, ħɯ)\big), \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big)\Big\}. \qquad (7.2.24)
\end{aligned}$$

Case 1. If $\max\Big\{\phi\big(ȡ(ɯ, ħɯ)\big), \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big)\Big\} = \phi\big(ȡ(ɯ, ħɯ)\big)$, then by (7.2.24), we have

$$\psi\big(ȡ(ħɯ, ɯ)\big) \leq \phi\big(ȡ(ɯ, ħɯ)\big) < \psi\big(ȡ(ħɯ, ɯ)\big), \text{ a contradiction.}$$

Case 2. If $\max\Big\{\phi\big(ȡ(ɯ, ħɯ)\big), \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big)\Big\} = \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big)$, then by (7.2.24), we have

$$\psi\big(ȡ(ħɯ, ɯ)\big) \leq \phi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big) < \psi\Big(\tfrac{ȡ(ɯ, ħɯ)}{2}\Big),$$

then, by monotone property of ψ, we have $ȡ(ħɯ, ɯ) < \frac{ȡ(ɯ, ħɯ)}{2}$, a contradiction.

Hence, $\text{ɯ} = \hbar\text{ɯ}$.

We now give an example in support of Theorem 7.2.2 as follows:

Example 7.2.1 Let $\mathfrak{X} = \mathbb{R}^+$ be endowed with the Euclidean metric d (say) and the partial order $\preccurlyeq$ be given by: $\varkappa \preccurlyeq y \Leftrightarrow (\varkappa = y)$ or $(\varkappa, y \geq 1, \varkappa \leq y)$.

Let $\hbar: \mathfrak{X} \to \mathfrak{X}$ be defined by $\hbar\varkappa = \begin{cases} \varkappa/2, & \text{if } 0 \leq \varkappa < 1 \\ 0, & \text{if } \varkappa \geq 1 \end{cases}$.

Let $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(\mathfrak{t}) = 3\mathfrak{t}$ and $\psi(\mathfrak{t}) = 4\mathfrak{t}$, respectively.

Take $\varkappa \preccurlyeq y$ and $\varkappa \neq y$, so that we have $1 \leq \varkappa < y$. Hence, we have $d(\hbar\varkappa, \hbar y) = 0$, so that $\psi(d(\hbar\varkappa, \hbar y)) = 0$ and $M(\varkappa, y) = 3y$. This implies that (7.2.1) holds for $\varkappa, y \in \mathfrak{X}$ with $\varkappa \preccurlyeq y$. Further, the Assumption 2.1.2 also holds in $\mathfrak{X}$. Also, the other conditions of Theorem 7.2.2 are satisfied and $\text{ɯ} = 0$ is a fixed point of $\hbar$.

Uniqueness Of Fixed Points

Next, we discuss a sufficient condition to obtain the uniqueness of the fixed point for the above proved results. For, we use the concept of **diameter** of a subset A of a metric space $(\mathfrak{X}, d)$ which is defined by

$$\operatorname{diam}(A) = \sup\{d(\varkappa, y): \varkappa, y \in A\}.$$

Now, we give our result as follows:

Theorem 7.2.3. Adding to the hypotheses of Theorem 7.2.1 (and Theorem 7.2.2) the following condition:

$$\lim_{n\to\infty} \operatorname{diam}(\hbar^n \mathfrak{X}) = 0, \tag{7.2.25}$$

we obtain the uniqueness of the fixed point of $\hbar$.

Proof. Let ɯ and v be two fixed points of $\hbar$, then $\text{ɯ} = \hbar\text{ɯ}$ and $\text{v} = \hbar\text{v}$.

It is easy to obtain for all $n \in \mathbb{N}$, that $\hbar^n\varkappa = \varkappa$, for $\varkappa \in \{\text{ɯ}, \text{v}\}$. Then, we have $d(\text{ɯ}, \text{v}) = d(\hbar^n\text{ɯ}, \hbar^n\text{v}) \leq \operatorname{diam}(\hbar^n\mathfrak{X}) \to 0$ as $n \to \infty$.Therefore, $\text{ɯ} = \text{v}$, which is to be proved.

In order to obtain the uniqueness of the fixed points of the self mappings, various authors (see, [40], [41], [53]) assumed the following assumption on $\mathfrak{X}$:

Assumption 7.2.1 ([40, 41]). "For all $(\varkappa, y) \in \mathfrak{X} \times \mathfrak{X}$, there exists a $\mathfrak{z} \in \mathfrak{X}$ such that $\varkappa \preccurlyeq \mathfrak{z}$ and $y \preccurlyeq \mathfrak{z}$".

Interestingly, Assumption 7.2.1 is not always applicable. We next formulate an example to obtain the uniqueness of the fixed points under condition (7.2.25) such that Assumption 7.2.1 does not hold.

Example 7.2.2. Let $\mathfrak{X} = \{3, 4, 5\}$ be endowed with the usual metric $\mathfrak{d}(\varkappa, y) = |\varkappa - y|$ for all $\varkappa$, $y \in \mathfrak{X}$ and the partial order be given by $\preccurlyeq\ := \{(3, 3), (4, 4), (5, 5), (5, 3)\}$. Consider $\hbar = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 5 & 3 \end{pmatrix}$.

Let $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$ and $\psi(\mathfrak{t}) = 2\mathfrak{t}$, respectively.

Now, we show that all the hypotheses of Theorem 7.2.2 are satisfied.

First, we show that $\mathfrak{X}$ satisfies Assumption 2.1.2. For, let $\{z_n\}$ be a non-decreasing sequence in $\mathfrak{X}$ w.r.t. $\preccurlyeq$ such that $z_n \to z$ ($\in \mathfrak{X}$) as $n \to \infty$. Now,

(i) if $z_0 = 3$, then $z_0 = 3 \preccurlyeq z_1$. Using the definition of $\preccurlyeq$, we get $z_1 = 3$. Applying induction, we get $z_n = 3$ for all $n \in \mathbb{N}$ and $z = 3$. Then, $z_n \preccurlyeq z$ for all $n \in \mathbb{N}$;

(ii) if $z_0 = 4$, then $z_0 = 4 \preccurlyeq z_1$. Using the definition of $\preccurlyeq$, we get $z_1 = 4$. Applying induction, we get $z_n = 4$ for all $n \in \mathbb{N}$ and $z = 4$. Then, $z_n \preccurlyeq z$ for all $n \in \mathbb{N}$;

(iii) if $z_0 = 5$, then $z_0 = 5 \preccurlyeq z_1$. Using the definition of $\preccurlyeq$, we get $z_1 = 5$ or 3. Applying induction, we get $z_n = 5$ or 3 for all $n \in \mathbb{N}$. Let there exists $p \geq 1$ such that $z_p = 3$. Now, using the definition of $\preccurlyeq$, we get $z_n = z_p = 3$ for all $n \geq p$. Therefore, we have $z = 3$and $z_n \preccurlyeq z$ for all $n \in \mathbb{N}$.

Further, the condition (7.2.1) also holds.

For, let $\varkappa$, $y \in \mathfrak{X}$ such that $\varkappa \preccurlyeq y$ and $\varkappa \neq y$, then, we have $\varkappa = 5$ and $y = 3$. In particular $\mathfrak{d}(\hbar 5, \hbar 3) = 0$, so that $\psi(\mathfrak{d}(\varkappa, y)) = 0$ and $\mathfrak{M}(\varkappa, y) = 1$. Thus, (7.2.1) holds easily. Also, $\hbar$ is non-decreasing mapping w.r.t. $\preccurlyeq$ and there exists $\varkappa_0 = 5$ such that $\varkappa_0 \preccurlyeq \hbar\varkappa_0$. Therefore, all the hypotheses of Theorem 7.2.2 are satisfied. Also, $\lim_{n\to\infty} \operatorname{diam}(\hbar^n \mathfrak{X}) = 0$. Clearly, $\mathfrak{u} = 3$ is the unique fixed point of the mapping $\hbar$. Further, we notice that for $(3, 4) \in \mathfrak{X} \times \mathfrak{X}$, there exists no $\mathfrak{z} \in \mathfrak{X}$ for which Assumption 7.2.1 holds.

The following results are the immediate consequences of Theorems 7.2.1 and 7.2.2.

Corollary 7.2.1. Let $(\mathfrak{X}, \preccurlyeq, \mathfrak{d})$ be a POCMS and $\hbar$ be a non-decreasing self-mapping on $\mathfrak{X}$ such that for all $\varkappa$, $y \in \mathfrak{X}$ with $\varkappa \succcurlyeq y$, we have

$$\psi\big(\mathfrak{d}(\hbar\varkappa, \hbar y)\big) \leq \max\{\phi\big(\mathfrak{d}(\varkappa, y)\big), \phi\big(\mathfrak{d}(\varkappa, \hbar\varkappa)\big), \phi\big(\mathfrak{d}(y, \hbar y)\big)\}, \tag{7.2.26}$$

where ψ is an ADF and $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for all $\mathfrak{t} > 0$. Suppose either

(a) ħ is continuous, or (b) Assumption 2.1.2 holds in Ӽ.

If there exists $\varkappa_0 \in$ Ӽ such that $\varkappa_0 \preccurlyeq ħ\varkappa_0$, then ħ has a fixed point. Also, if Ӽ satisfies condition (7.2.25), we obtain the uniqueness of the fixed point.

Corollary 7.2.2 ([171]). Let (Ӽ, ≼, ȡ) be a POCMS and ħ be a non-decreasing self-mapping on Ӽ such that for all $\varkappa$, y ∈ Ӽ with $\varkappa \succcurlyeq$ y, we have

$$\psi\big(ȡ(ħ\varkappa, ħy)\big) \leq \phi\big(ȡ(\varkappa, y)\big), \tag{7.2.27}$$

where ψ is an ADF and ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for all ŧ > 0. Suppose either

(a) ħ is continuous, or (b) Assumption 2.1.2 holds in Ӽ.

If there exists $\varkappa_0 \in$ Ӽ such that $\varkappa_0 \preccurlyeq ħ\varkappa_0$, then ħ has a fixed point in Ӽ.

On adding the condition (7.2.25) in Corollary 7.2.2, we obtain the uniqueness of the obtained fixed point in Corollary 7.2.2.

Remark 7.2.1. (i) Considering ψ to be the identity function and $\phi(\varkappa) = \varkappa - \psi(\varkappa)$ in Corollary 7.2.2, the condition (7.2.27) becomes (7.1.2) (which is due to Harjani and Sadarangani [47]).

(ii) On taking $\phi(\varkappa)$ to be $\psi(\varkappa) - \phi_1(\varkappa)$ in Corollary 7.2.2, the condition (7.2.27) becomes (7.1.3) (which is due to Harjani and Sadarangani [48]), where ϕ_1 is an ADF.

(iii) By defining ψ to be the identity function and $\phi(\varkappa) = \beta(\varkappa)\varkappa$ in Corollary 7.2.2, the condition (7.2.27) transforms into (7.1.4) (which is due to Amini-Harandi and Emami [53]), where $\beta \in \mathfrak{R}$.

(iv) On taking ψ to be the identity function and $\phi(\varkappa)$ = ķ$\varkappa$ (where ķ ∈ (0, 1)) in Corollary 7.2.2, the condition (7.2.27) becomes (7.1.1) (which is due to Ran and Reurings [40], Nieto and L$\acute{o}$pez [41]).

Coincidence And Common Fixed Points

Now, we generalize the notion of generalized weak ($\psi > \phi$) – contraction for the pair of self mappings as follows:

Definition 7.2.2. Let (Ӽ, ȡ) be a metric space and ħ, ꞡ be two self mappings on Ӽ. The mapping ħ is called a **generalized weak ($\boldsymbol{\psi > \phi}$) – contraction w.r.t. ꞡ** if it satisfies the following condition:

$$\psi\big(ȡ(ħ\varkappa, ħy)\big) \leq M_{ꞡ}(\varkappa, y), \tag{7.2.28}$$

for all $\varkappa$, y ∈ Ӽ, where ψ is an ADF and ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that ψ(ŧ) > ϕ(ŧ) for all ŧ > 0 and

$$M_g(\varkappa, y) = \max\left\{\phi\big(d(g\varkappa, gy)\big), \phi\big(d(g\varkappa, \hbar x)\big), \phi\big(d(gy, \hbar y)\big), \phi\left(\frac{d(g\varkappa, \hbar y)+d(gy, \hbar\varkappa)}{2}\right)\right\}. \tag{7.2.29}$$

Theorem 7.2.4. Let $(X, \preccurlyeq, d)$ be a POMS and $\hbar$, g be the two self-mappings on X, where $\hbar$ is a g-non-decreasing mapping such that $\hbar$ is a generalized weak $(\psi > \phi)$ – contraction mapping w.r.t. g for all $\varkappa, y \in X$ with $g\varkappa \succcurlyeq gy$. Also, suppose that $g(X)$ is a complete subspace of X and $\hbar(X) \subseteq g(X)$. Suppose either

(a) $\hbar$ and g are both continuous, or (b) Assumption 2.1.2 holds in X.

If there exists $\varkappa_0 \in X$ such that $g\varkappa_0 \preccurlyeq \hbar\varkappa_0$, then $\hbar$ and g have a coincidence point in X.

Proof. By Lemma 7.1.1, there exists $A \subseteq X$ such that $g(A) = g(X)$ and $g: A \to X$ is one-to-one. Define a mapping $f: g(A) \to g(A)$ by

$$fg\varkappa = \hbar x, \text{ for } g\varkappa \in g(A) \tag{7.2.30}$$

Since g is one-to-one on A, so f is well-defined. Also, we have

$$\psi\big(d(\hbar\varkappa, \hbar y)\big)$$
$$\leq \max\left\{\phi\big(d(g\varkappa, gy)\big), \phi\big(d(g\varkappa, \hbar x)\big), \phi\big(d(gy, \hbar y)\big), \phi\left(\frac{d(g\varkappa, \hbar y)+d(gy, \hbar\varkappa)}{2}\right)\right\},$$

for all $g\varkappa, gy \in g(A)$ with $g\varkappa \succcurlyeq gy$. Since $\hbar$ is a g-non-decreasing mapping, for all $g\varkappa_1, g\varkappa_2 \in g(A)$, $g\varkappa_1 \preccurlyeq g\varkappa_2$ implies $\hbar\varkappa_1 \preccurlyeq \hbar\varkappa_2$, so that we have $fg\varkappa_1 \preccurlyeq fg\varkappa_2$ which implies that f is a non-decreasing mapping. Also there exists $\varkappa_0 \in X$ such that $g\varkappa_0 \preccurlyeq \hbar\varkappa_0$, which implies the existence of $g\varkappa_0 \in g(X)$ such that $g\varkappa_0 \preccurlyeq fg\varkappa_0$.

Assume that assumption (a) holds. Since both $\hbar$ and g are continuous, so f is also continuous. On applying Theorem 7.2.1 to the mapping f, we can obtain that f has a fixed point u (say) in $g(X)$.

Assume that assumption (b) holds. Then as above, on applying Theorem 7.2.2, we can conclude that f has a fixed point u (say) in $g(X)$.

Finally, we show that $\hbar$ and g have a coincidence point. Since u is a fixed point of f, we have

$$u = fu. \tag{7.2.31}$$

Also, since $u \in g(X)$, there exists a point $u_0 \in X$ such that

$$u = gu_0. \tag{7.2.32}$$

Now, using (7.2.31) and (7.2.32), we have

$$gu_0 = fgu_0. \tag{7.2.33}$$

By (7.2.30) and (7.2.33), we can obtain that $gu_0 = \hbar u_0$.

Therefore, $ư_0$ is a coincidence point of ħ and g.

The following result is an immediate consequence of Theorem 7.2.4:

Corollary 7.2.3. Let (Ҳ, ≼, ḏ) be a POMS and ħ, g be the two self-mappings on Ҳ, where ħ is a g-non-decreasing mapping such that

$$\psi\big(ḏ(ħ\varkappa, ħy)\big) \le \phi\big(ḏ(g\varkappa, gy)\big), \tag{7.2.34}$$

for all $\varkappa \succcurlyeq$ y, where ψ is an ADF and ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for all ŧ > 0.

Assume that g(Ҳ) is a complete subspace of Ҳ and ħ(Ҳ) ⊆ g(Ҳ). Suppose either

(a) ħ and g are both continuous, or (b) Assumption 2.1.2 holds in Ҳ.

If there exists $\varkappa_0$ ∈ Ҳ such that $g\varkappa_0 \preccurlyeq ħ\varkappa_0$, then ħ and g have a coincidence point in Ҳ.

Remark 7.2.2. In Theorem 7.2.4 (and Corollary 7.2.3) if the mappings ħ and g are weakly compatible, then they have a common fixed point in Ҳ. Then, the result obtained from Theorem 7.2.4 generalizes the corresponding result of *Ćirić* et al. [52], that is Theorem 2.1.12, since the contraction (7.2.28) generalizes the contraction (7.1.5).

Theorem 7.2.5. Adding to the hypotheses of Theorem 7.2.4 (and Corollary 7.2.3) the following conditions:

(i) the pair of mappings (ħ, g) is weakly compatible;

(ii) $\lim_{n\to\infty} \operatorname{diam}((ħ\circ g)^n Ҳ) = 0$,

(where ∘ denotes the composition of mappings), we obtain the uniqueness of the fixed point of ħ and g.

Proof. Let ư and ν be two common fixed points of ħ and g, that is,

ư = ħư = gư and ν = ħν = gν.

It is immediate to show that for all n ∈ ℕ, we have:

$(ħ\circ g)^n \varkappa = \varkappa$, for all $\varkappa$ ∈ {ư, ν}.

Then, we have

$$ḏ(ư, ν) = ḏ((ħ\circ g)^n ư, (ħ\circ g)^n ν) \le \operatorname{diam}((ħ\circ g)^n Ҳ) \to 0 \text{ as } n \to \infty.$$

Therefore ư = ν.

7.3. APPLICATION OF GENERALIZED WEAK $(\psi > \phi)$ – CONTRACTIONS TO COUPLED FIXED POINT PROBLEMS

Using the approach of Samet et al. [172], as an application of the results obtained in section 7.2, we establish some coupled fixed point theorems that also generalize many coupled fixed point results present in the literature. For this, we need to consider the following:

Let (Ӿ, ≼, ɖ) be a POMS and Ғ: Ӿ × Ӿ → Ӿ be a given mapping. Endow the set ϒ = Ӿ × Ӿ with the partial order ⊑ given as:

$$(\varkappa, y) \sqsubseteq (u, v) \Leftrightarrow \varkappa \preccurlyeq u,\ y \succcurlyeq v, \text{ for } (\varkappa, y), (u, v) \in \Upsilon.$$

Further, the mappings η, δ: ϒ → $\mathbb{R}^+$ defined respectively by

$$\eta((\varkappa, y), (u, v)) = ɖ(\varkappa, u) + ɖ(y, v) \text{ and } \delta((\varkappa, y), (u, v)) = \max\{ɖ(\varkappa, u), ɖ(y, v)\}$$

for (ϰ, y), (u, v) ∈ ϒ, are metrics on ϒ.

Also, define a mapping τ: ϒ → ϒ by

$$\tau(\varkappa, y) = (Ғ(\varkappa, y), Ғ(y, \varkappa)) \text{ for all } (\varkappa, y) \in \Upsilon".$$

Lemma 7.3.1 ([172]). The following properties hold:

(a) (Ӿ, ɖ) is complete iff (ϒ, η) and (ϒ, δ) are complete;

(b) Ғ has MMP iff τ is monotone non-decreasing w.r.t. ⊑;

(c) (ϰ, y) ∈ ϒ is a coupled fixed point of Ғ iff (ϰ, y) is a fixed point of τ.

Before we proceed, let us recall some notions useful in our results.

Assumption 2.1.7 ([55]). Ӿ has the property:

(i) "if a non-decreasing sequence $\{\varkappa_n\}_{n=0}^{\infty} \subset$ Ӿ converges to ϰ, then $\varkappa_n \preccurlyeq \varkappa$ for all n";

(ii) "if a non-increasing sequence $\{y_n\}_{n=0}^{\infty} \subset$ Ӿ converges to y, then $y \preccurlyeq y_n$ for all n".

Property (P1): "There exist $\varkappa_0, y_0 \in$ Ӿ such that $\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ғ(y_0, \varkappa_0)$".

Property (P2): "There exist $\varkappa_0$, $y_0 \in$ Ӿ such that $g\varkappa_0 \preccurlyeq Ғ(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq Ғ(y_0, \varkappa_0)$".

Now, we formulate our results:

Theorem 7.3.1. Let (Ӿ, ≼, ɖ) be a POCMS and Ғ: Ӿ × Ӿ → Ӿ be the mapping with MMP on Ӿ. Assume that the following condition holds for all (ϰ, y), (u, v) ∈ Ӿ × Ӿ with ϰ ≽ u and y ≼ v,

$$\psi\left(ɖ\left(Ғ(\varkappa, y), Ғ(u, v)\right)\right) \le \phi(\max\{ɖ(\varkappa, u), ɖ(y, v)\}), \tag{7.3.1}$$

where ψ is an ADF and ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(\mathfrak{f}) > \phi(\mathfrak{f})$ for all $\mathfrak{f} > 0$. Suppose either

(a) Ꞙ is continuous, or (b) Ӽ assumes Assumption 2.1.7.

If Ӽ has the property (P1), then Ꞙ has a coupled fixed point in Ӽ.

Proof. By (7.3.1), for all $(\varkappa, y), (u, v) \in Ӽ \times Ӽ$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we have

$$\psi\left(d(Ꞙ(\varkappa, y), Ꞙ(u, v))\right) \leq \phi(\max\{d(\varkappa, u), d(y,v)\}) \quad (7.3.2)$$

and $$\psi\left(d(Ꞙ(v, u), Ꞙ(y, \varkappa))\right) \leq \phi(\max\{d(\varkappa, u), d(y,v)\}). \quad (7.3.3)$$

As the function ψ is non-decreasing, then, for all $(\varkappa, y), (u, v) \in Ӽ \times Ӽ$ with $\varkappa \succcurlyeq u$ and $y \preccurlyeq v$, we have

$$\psi\left(\max\{d(Ꞙ(\varkappa, y), Ꞙ(u, v)), d(Ꞙ(v, u), Ꞙ(y, \varkappa))\}\right) \leq \phi(\max\{d(\varkappa, u), d(y,v)\}),$$

that is, $\psi\left(\delta(\tau(\varkappa, y), \tau(u,v))\right) \leq \phi\left(\delta((\varkappa, y), (u,v))\right)$, for all $(\varkappa, y), (u, v) \in \Upsilon$ with $(\varkappa, y) \sqsupseteq (u, v)$.

Then using Lemma 7.3.1, we note the followings:

(i) "completeness of (Ӽ, d) implies the completeness of (Υ, δ)";

(ii) "MMP of mapping Ꞙ in Ӽ implies that mapping τ is non-decreasing w.r.t. $\sqsubseteq$".

By (P1), "there exist $\varkappa_0, y_0 \in Ӽ$ such that $\varkappa_0 \preccurlyeq Ꞙ(\varkappa_0, y_0)$ and $y_0 \succcurlyeq Ꞙ(y_0, \varkappa_0)$", so that we have $(\varkappa_0, y_0) \sqsubseteq \tau(\varkappa_0, y_0)$.

Assume that assumption (a) holds, so that Ꞙ is continuous, and hence, the τ is also continuous. Now, applying Theorem 7.2.1 we can obtain that τ has a fixed point, which in turn, on using Lemma 7.3.1, implies that Ꞙ has a coupled fixed point.

Assume that assumption (b) holds, so that Ӽ assumes Assumption 2.1.7, then we can easily obtain that: "if a non-decreasing (w.r.t. $\sqsubseteq$) sequence $\{u_n\}$ in Υ converges to some point $u \in \Upsilon$, then $u_n \sqsubseteq u$ for all n".

Now, in both the cases, on applying Corollary 7.2.2, we can get that τ has a fixed point, which in turn implies that Ꞙ has a coupled fixed point.

Remark 7.3.1. (i) For $\phi(\varkappa) = \psi(\varkappa) - \phi_1(\varkappa)$, contraction (7.3.1) becomes (7.1.7) (where ϕ_1 is ADF taken by Harjani et al. [58]). Therefore, result of Harjani et al. [58] (Theorem 2.1.15) is a particular case of Theorem 7.3.1.

(ii) The inequality $d(Ꞙ(\varkappa, y), Ꞙ(u, v)) \leq \frac{k}{2}[d(\varkappa, u) + d(y,v)]$ (that is, condition (7.1.6)) is contained in $d(Ꞙ(\varkappa, y), Ꞙ(u, v)) \leq k \max\{d(\varkappa, u), d(y,v)\}$, where $0 \leq k < 1$, which is actually the condition (7.3.1) with ψ being the identity function and $\phi(\varkappa)$

$= k\varkappa$, $k \in [0,1)$. Therefore, the results of Bhaskar and Lakshmikantham [55] (Theorem 2.1.14 along with the Assumption 2.1.7) is a special case of Theorem 7.3.1.

(iii) Result of Berinde [149], that is Theorem 3.1.1, is a particular case of Corollary 7.2.2 for $\psi(\varkappa) = \frac{\varkappa}{2}$, and $\phi(\varkappa) = \frac{k\varkappa}{2}$, $k \in [0,1)$ since we know that for all $\varkappa$, y, u, v $\in X$, the following Berinde's contractive condition (that is, the condition (7.1.8)):

$$d\big(F(\varkappa,y),F(u,v)\big) + d\big(F(y,\varkappa),F(v,u)\big) \le k[d(\varkappa,u) + d(y,v)]$$

can be expressed as $\quad \eta\big(\tau(\varkappa,y),\tau(u,v)\big) \le k\,\eta\big((\varkappa,y),(u,v)\big)$

or $\quad \psi\Big(\eta\big(\tau(\varkappa,y),\tau(u,v)\big)\Big) \le \phi\Big(\eta\big((\varkappa,y),(u,v)\big)\Big)$, for $(\varkappa, y), (u, v) \in X \times X = \Upsilon$.

(iv) For ψ to be the identity function and $\phi(\varkappa) = \beta(\varkappa)\,\varkappa$, where $\beta \in \Re$, contraction (7.3.1) becomes (7.1.9). Therefore, result of Rasouli and Bahrampour [70] (Theorem 2.1.23) is a special case of Theorem 7.3.1.

Theorem 7.3.2. Let $(X, \preccurlyeq, d)$ be a POMS and $F: X \times X \to X$, $g: X \to X$ be the mappings such that F has MgMP on X. Assume that g(X) is a complete subspace of X and $F(X \times X) \subseteq g(X)$. Suppose that

$$\psi\Big(d\big(F(\varkappa,y),F(u,v)\big)\Big) \le \phi(\max\{d(g\varkappa,gu),d(gy,gv)\}), \tag{7.3.4}$$

for all $(\varkappa, y), (u, v) \in X \times X$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$, where ψ is an ADF and $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(t) > \phi(t)$ for all $t > 0$.

Suppose either

(a) F and g both are continuous, or (b) X assumes Assumption 2.1.7.

If X has the property (P2), then F and g have a coupled coincidence point in X.

Proof. By Lemma 7.1.1, there exists a subset $A \subseteq X$ such that $g(A) = g(X)$ and the mapping $g: A \to X$ is one-to-one. Define a mapping $H: g(A) \times g(A) \to X$ by

$$H(ga, gb) = F(a, b) \tag{7.3.5}$$

for all $ga, gb \in g(A) = g(X)$.

Since g is one-to-one on A, so H is well-defined. By (7.3.4) and (7.3.5), it follows that

$$\psi\Big(d\big(H(g\varkappa,gy),H(gu,gv)\big)\Big) = \psi\Big(d\big(F(\varkappa,y),F(u,v)\big)\Big)$$
$$\le \phi(\max\{d(g\varkappa,gu),d(gy,gv)\})$$

for all $g\varkappa, gy, gu, gv \in g(X)$ with $g\varkappa \succcurlyeq gu$ and $gy \preccurlyeq gv$. As F has the MgMP, for all $g\varkappa, gy \in g(X)$, we have

$$g\varkappa_1, g\varkappa_2 \in g(X), \quad g\varkappa_1 \preccurlyeq g\varkappa_2 \text{ implies } H(g\varkappa_1, gy) \preccurlyeq H(g\varkappa_2, gy), \tag{7.3.6}$$

$$gy_1, gy_2 \in g(X), \quad gy_1 \preccurlyeq gy_2 \text{ implies } H(g\varkappa, gy_1) \succcurlyeq H(g\varkappa, gy_2), \tag{7.3.7}$$

which implies that Ή has MMP. Also, there exist $\varkappa_0$, $y_0 \in X$ such that $g\varkappa_0 \preccurlyeq F(\varkappa_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, \varkappa_0)$, which implies the existence of $g\varkappa_0$, $gy_0 \in X$ such that $g\varkappa_0 \preccurlyeq Ή(g\varkappa_0, gy_0)$ and $gy_0 \succcurlyeq Ή(gy_0, g\varkappa_0)$.

Suppose that assumption (a) holds, that is, both F and g are continuous. Then, the continuity of F and g implies the continuity of Ή. Applying Theorem 7.3.1 for the mapping Ή, we can obtain that Ή has a coupled fixed point in $g(X) \times g(X)$, say (u, v).

Suppose that the assumption (b) holds. Then, by Theorem 7.3.1, similarly we can conclude that Ή has a coupled fixed point in $g(X) \times g(X)$, say (u, v).

Now, finally, in both the cases, we show that F and g have a coupled coincidence point. If (u, v) is a coupled fixed point of Ή, we have

$$u = Ή(u, v) \text{ and } v = Ή(v, u). \tag{7.3.8}$$

As $(u, v) \in g(X) \times g(X)$, there exists some $(u_0, v_0) \in X \times X$ such that

$$u = gu_0 \text{ and } v = gv_0. \tag{7.3.9}$$

Then, using (7.3.8) and (7.3.9) we get

$$gu_0 = Ή(gu_0, gv_0) \text{ and } gv_0 = Ή(gv_0, gu_0). \tag{7.3.10}$$

Now, using (7.3.5) and (7.3.10) we can obtain that

$$gu_0 = F(u_0, v_0) \text{ and } gv_0 = F(v_0, u_0). \tag{7.3.11}$$

Therefore, (u_0, v_0) is a required coupled coincidence point of F and g.

Remark 7.3.2. (i) For $\phi(\varkappa) = \psi(\varkappa) - \phi'(\varkappa)$ the contraction (7.3.4) becomes (7.1.10) (where ϕ' is an ADF taken by Choudhury et al. [56]). Therefore, Theorem 7.3.2 generalizes the recent result of Choudhury et al. [56] (Theorem 2.1.18).

(ii) Result of Jain et al. [159], that is Corollary 3.2.1, is a particular case of Corollary 7.2.3 for $\psi(\varkappa) = \frac{\varkappa}{2}$ and $\phi(\varkappa) = \frac{k\varkappa}{2}$, $k \in [0, 1)$, since we know that for all $\varkappa$, y, u, v $\in X$, the following contractive condition (which is the contractive condition (7.1.11)):

$$d\big(F(\varkappa, y), F(u, v)\big) + d\big(F(y, \varkappa), F(v, u)\big) \leq k\,[d(g\varkappa, gu) + d(gy, gv)]$$

can be expressed as

$$\eta\big(\tau(\varkappa, y), \tau(u,v)\big) \leq k\,\eta\big((g\varkappa, gy), (gu, gv)\big)$$

or $$\psi\Big(\eta\big(\tau(\varkappa, y), \tau(u,v)\big)\Big) \leq \phi\Big(\eta\big((g\varkappa, gy), (gu, gv)\big)\Big),$$

for $(\varkappa, y), (u, v) \in X \times X = \Upsilon$.

7.4. NEW GENERALIZED NONLINEAR CONTRACTIVE CONDITION IN COUPLED FIXED POINT THEORY

Recently, *Đ*oric et al. [173] replaced MMP by another property which is satisfied automatically in totally ordered spaces. In this section, using this property, we establish some results under new generalized nonlinear contractive conditions in coupled fixed point theory. The work presented in this section generalize the results of Bhaskar and Lakshmikantham [55], Harjani et al. [58], Rasouli and Bahrampour [70], Choudhury et al. [56], Luong and Thuan [69], Karapinar et al. [57] and Chandok and Tas [174].

We consider the following notations some of which are due to *Đ*oric et al. [173]: "If elements $\varkappa$, y of a poset (Ӽ, $\preccurlyeq$) are comparable (that is, $\varkappa \preccurlyeq y$ or $y \preccurlyeq \varkappa$ holds), we shall write $\varkappa \asymp y$".

Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be two mappings. We shall consider the following condition:

"if $\varkappa$, y, u ∈ Ӽ are such that $g\varkappa \asymp Ғ(\varkappa, y) = gu$, then $Ғ(\varkappa, y) \asymp Ғ(u, v)$ for v ∈ Ӽ". (7.4.1)

In particular, for g being the identity mapping on Ӽ, (7.4.1) reduces to

"for all $\varkappa$, y ∈ Ӽ, if $\varkappa \asymp Ғ(\varkappa, y)$, then $Ғ(\varkappa, y) \asymp Ғ(Ғ(\varkappa, y), v)$ for v ∈ Ӽ". (7.4.2)

In our results, we also use the following assumption:

Assumption 7.4.1 ([173]). Ӽ has the property: "$\varkappa_n \to \varkappa$, when $n \to \infty$ in Ӽ, then $\varkappa_n \asymp \varkappa$ for sufficiently large n".

Now, we give our results as follows:

Theorem 7.4.1. Let (Ӽ, $\preccurlyeq$, ȡ) be a POMS and Ғ: Ӽ × Ӽ → Ӽ, g: Ӽ → Ӽ be two mappings. Assume that g(Ӽ) is complete, Ғ(Ӽ × Ӽ) ⊆ g(Ӽ), g and Ғ satisfy the condition (7.4.1) and there exists some Ł ≥ 0 such that

$$\psi\Big(ȡ\big(Ғ(\varkappa, y), Ғ(u, v)\big)\Big) \le \phi(\max\{ȡ(g\varkappa, gu), ȡ(gy, gv)\}) + Ł \min\left\{\begin{matrix} ȡ(Ғ(\varkappa, y), gu), ȡ(Ғ(u, v), g\varkappa), \\ ȡ(Ғ(\varkappa, y), g\varkappa), ȡ(Ғ(u, v), gu) \end{matrix}\right\}, \quad (7.4.3)$$

for all $\varkappa$, y, u, v ∈ Ӽ with $g\varkappa \asymp gu$ and $gy \asymp gv$, where ψ is an ADF and $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\psi(f) > \phi(f)$ for all f > 0. Also, suppose either

(a) Ғ and g both are continuous, or (b) Ӽ assumes Assumption 7.4.1.

Suppose Ӽ has the following property:

(P9) "there exist $\varkappa_0, y_0$ ∈ Ӽ such that $g\varkappa_0 \asymp Ғ(\varkappa_0, y_0)$ and $gy_0 \asymp Ғ(y_0, \varkappa_0)$".

Then, F and g have a coupled coincidence point in X.

Proof. Since X has the property (P9), there exist $\varkappa_0, y_0 \in \mathrm{X}$ such that $g\varkappa_0 \asymp \mathrm{F}(\varkappa_0, y_0)$ and $gy_0 \asymp \mathrm{F}(y_0, \varkappa_0)$. Now, since $\mathrm{F}(\mathrm{X} \times \mathrm{X}) \subseteq g(\mathrm{X})$, sequences $\{\varkappa_n\}$ and $\{y_n\}$ can be constructed in X such that $g\varkappa_{n+1} = \mathrm{F}(\varkappa_n, y_n)$ and $gy_{n+1} = \mathrm{F}(y_n, \varkappa_n)$, for $n \in \mathbb{N}$.

Again using (P9), $g\varkappa_0 \asymp \mathrm{F}(\varkappa_0, y_0) = g\varkappa_1$ and $gy_0 \asymp \mathrm{F}(y_0, \varkappa_0) = gy_1$, then since g and F satisfies (7.4.1), we obtain $g\varkappa_1 = \mathrm{F}(\varkappa_0, y_0) \asymp \mathrm{F}(\varkappa_1, y_1) = g\varkappa_2$ and $gy_1 = \mathrm{F}(y_0, \varkappa_0) \asymp \mathrm{F}(y_1, \varkappa_1) = gy_2$. Then, inductively, we can obtain that $g\varkappa_{n-1} \asymp g\varkappa_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$.

Now using (7.4.3), we have

$$\begin{aligned}\psi\big(d(g\varkappa_{n+1}, g\varkappa_n)\big) = \psi\Big(d\big(\mathrm{F}(\varkappa_n, y_n), \mathrm{F}(\varkappa_{n-1}, y_{n-1})\big)\Big) &\leq \phi(\max\{d(g\varkappa_n, g\varkappa_{n-1}), d(gy_n, gy_{n-1})\}) \\ &\quad + \mathrm{Ł}\min\left\{\begin{matrix} d(\mathrm{F}(\varkappa_n, y_n), g\varkappa_{n-1}), d(\mathrm{F}(\varkappa_{n-1}, y_{n-1}), g\varkappa_n), \\ d(\mathrm{F}(\varkappa_n, y_n), g\varkappa_n), d(\mathrm{F}(\varkappa_{n-1}, y_{n-1}), g\varkappa_{n-1}) \end{matrix}\right\}, \quad (7.4.4)\end{aligned}$$

which implies that

$$\psi\big(d(g\varkappa_{n+1}, g\varkappa_n)\big) \leq \phi(\max\{d(g\varkappa_n, g\varkappa_{n-1}), d(gy_n, gy_{n-1})\}). \quad (7.4.5)$$

Similarly, we can get

$$\psi\big(d(gy_{n+1}, gy_n)\big) \leq \phi(\max\{d(gy_n, gy_{n-1}), d(g\varkappa_n, g\varkappa_{n-1})\}). \quad (7.4.6)$$

Since $\max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\}$ is either $d(g\varkappa_{n+1}, g\varkappa_n)$ or $d(gy_{n+1}, gy_n)$, in both the cases, by (7.4.5) and (7.4.6) we can get

$$\begin{aligned}&\psi(\max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\}) \\ &\qquad \leq \phi(\max\{d(g\varkappa_n, g\varkappa_{n-1}), d(gy_n, gy_{n-1})\}). \quad (7.4.7)\end{aligned}$$

Now, since $\psi(\mathrm{t}) > \phi(\mathrm{t})$ for all $\mathrm{t} > 0$, we obtain that

$$\max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\} \leq \max\{d(g\varkappa_n, g\varkappa_{n-1}), d(gy_n, gy_{n-1})\}.$$

Denote $\mathrm{R}_n = \max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\}$, then $\{\mathrm{R}_n\}$ is a non-increasing sequence of positive real numbers. So, there exists some $\mathrm{R} \geq 0$ such that $\lim_{n\to\infty} \mathrm{R}_n = \mathrm{R}$.

We claim that $\mathrm{R} = 0$.

On the contrary, assume that $\mathrm{R} > 0$. Now, taking $n \to \infty$ in (7.4.7) and using the properties of ψ and ϕ, we have

$$\begin{aligned}\psi(\mathrm{R}) &= \lim_{n\to\infty} \psi(\max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\}) \\ &\leq \lim_{n\to\infty} \phi(\max\{d(g\varkappa_n, g\varkappa_{n-1}), d(gy_n, gy_{n-1})\}) = \phi(\mathrm{R}) < \psi(\mathrm{R}), \quad (7.4.8)\end{aligned}$$

a contradiction. Therefore, $\mathrm{R} = 0$ and hence

$$\lim_{n\to\infty} \mathrm{R}_n = \lim_{n\to\infty} \max\{d(g\varkappa_{n+1}, g\varkappa_n), d(gy_{n+1}, gy_n)\} = 0. \quad (7.4.9)$$

Next, we claim that $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences. On the contrary, let at least one of $\{g\varkappa_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$,

$$d_k = \max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \geq \varepsilon. \tag{7.4.10}$$

Also, corresponding to m(k), we can choose n(k) in such a way that it is the smallest integer with $n(k) > m(k) > k$ and satisfying (7.4.10). Then, we have

$$\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)})\} < \varepsilon. \tag{7.4.11}$$

Also, using the triangle inequality and (7.4.11), we have

$$\begin{aligned} d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) &\leq d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}) + d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)}) \\ &< d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}) + \varepsilon, \end{aligned} \tag{7.4.12}$$

and

$$\begin{aligned} d(gy_{n(k)}, gy_{m(k)}) &\leq d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) \\ &< d(gy_{n(k)}, gy_{n(k)-1}) + \varepsilon. \end{aligned} \tag{7.4.13}$$

By (7.4.10), (7.4.12) and (7.4.13), we have

$$\begin{aligned} \varepsilon \leq d_k &= \max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &< \max\{d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} + \varepsilon. \end{aligned} \tag{7.4.14}$$

Taking $k \to \infty$ in (7.4.14) and using (7.4.9), we have

$$\lim_{k\to\infty} d_k = \lim_{k\to\infty} \max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} = \varepsilon. \tag{7.4.15}$$

Now, using the triangle inequality, we have

$$\begin{aligned} d(g\varkappa_{n(k)}, g\varkappa_{m(k)}) \leq d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}) &+ d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}) \\ &+ d(g\varkappa_{m(k)-1}, g\varkappa_{m(k)}), \end{aligned} \tag{7.4.16}$$

and

$$\begin{aligned} d(gy_{n(k)}, gy_{m(k)}) \leq d(gy_{n(k)}, gy_{n(k)-1}) &+ d(gy_{n(k)-1}, gy_{m(k)-1}) \\ &+ d(gy_{m(k)-1}, gy_{m(k)}). \end{aligned} \tag{7.4.17}$$

By (7.4.16) and (7.4.17), we have

$$\begin{aligned} \varepsilon \leq d_k &= \max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &\leq \max\{d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} \\ &\quad + \max\{d(g\varkappa_{m(k)-1}, g\varkappa_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\} \\ &\quad + \max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} \\ &= R_{n(k)-1} + R_{m(k)-1} + \max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}. \end{aligned} \tag{7.4.18}$$

Again, using triangle inequality and (7.4.11), we obtain that

$$d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}) \le d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)}) + d(g\varkappa_{m(k)}, g\varkappa_{m(k)-1})$$
$$< d(g\varkappa_{m(k)}, g\varkappa_{m(k)-1}) + \varepsilon,$$

and

$$d(gy_{n(k)-1}, gy_{m(k)-1}) \le d(gy_{n(k)-1}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)-1})$$
$$< d(gy_{m(k)}, gy_{m(k)-1}) + \varepsilon.$$

Therefore,

$$\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}$$
$$< \max\{d(g\varkappa_{m(k)}, g\varkappa_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\} + \varepsilon. \quad (7.4.19)$$

Now, on using (7.4.19) in (7.4.18), then taking the limit as $k \to \infty$ and using (7.4.9) and (7.4.15), we can obtain

$$\lim_{k\to\infty} \max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} = \varepsilon. \quad (7.4.20)$$

Since $g\varkappa_{n(k)-1} \asymp g\varkappa_{m(k)-1}$ and $gy_{n(k)-1} \asymp gy_{m(k)-1}$, then by (7.4.3), we obtain that

$$\psi\left(d(g\varkappa_{n(k)}, g\varkappa_{m(k)})\right)$$
$$= \psi\left(d\left(F(\varkappa_{n(k)-1}, y_{n(k)-1}), F(\varkappa_{m(k)-1}, y_{m(k)-1})\right)\right)$$
$$\le \phi(\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\})$$
$$+ Ł \min\left\{\begin{matrix} d(F(\varkappa_{n(k)-1}, y_{n(k)-1}), g\varkappa_{m(k)-1}), d(F(\varkappa_{m(k)-1}, y_{m(k)-1}), g\varkappa_{n(k)-1}), \\ d(F(\varkappa_{n(k)-1}, y_{n(k)-1}), g\varkappa_{n(k)-1}), d(F(\varkappa_{m(k)-1}, y_{m(k)-1}), g\varkappa_{m(k)-1}) \end{matrix}\right\}$$
$$\le \phi(\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\})$$
$$+ Ł \min\{d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}), d(g\varkappa_{m(k)}, g\varkappa_{m(k)-1})\}. \quad (7.4.21)$$

Similarly, we can obtain

$$\psi\left(d(gy_{n(k)}, gy_{m(k)})\right)$$
$$\le \phi(\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\})$$
$$+ Ł \min\{d(gy_{n(k)}, gy_{n(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\}. \quad (7.4.22)$$

Since $\max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}$ is either $d(g\varkappa_{n(k)}, g\varkappa_{m(k)})$ or $d(gy_{n(k)}, gy_{m(k)})$, using (7.4.21) and (7.4.22), we get

$$\psi(\max\{d(g\varkappa_{n(k)}, g\varkappa_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\})$$
$$\le \phi(\max\{d(g\varkappa_{n(k)-1}, g\varkappa_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\})$$
$$+ Ł \min\{d(g\varkappa_{n(k)}, g\varkappa_{n(k)-1}), d(g\varkappa_{m(k)}, g\varkappa_{m(k)-1})\}$$
$$+ Ł \min\{d(gy_{n(k)}, gy_{n(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\}. \quad (7.4.23)$$

Taking $k \to \infty$ in (7.4.23) and using (7.4.9), (7.4.15), (7.4.20) and the properties of ψ and ϕ, we have

$$\psi(\varepsilon) \leq \phi(\varepsilon) + 2\, Ł \min\{0, 0\} < \psi(\varepsilon),$$

a contradiction. Therefore, $\{g\varkappa_n\}$ and $\{gy_n\}$ are Cauchy sequences and hence, by completeness of g(X), there exist some $\varkappa$, y ∈ X such that

$$\begin{aligned} \lim_{n\to\infty} g\varkappa_n &= \lim_{n\to\infty} F(\varkappa_n, y_n) = g\varkappa, \\ \lim_{n\to\infty} gy_n &= \lim_{n\to\infty} F(y_n, \varkappa_n) = gy. \end{aligned} \tag{7.4.24}$$

Suppose that condition (a) holds.

Now, using Lemma 7.1.1, there exists a subset A ⊆ X such that g(A) = g(X) and the mapping g: A → X is one-to-one. Define a mapping H: g(A) × g(A) → X by

$$H(ga, gb) = F(a, b) \text{ for all } ga, gb \in g(A). \tag{7.4.25}$$

Since g is one-one, so H is well-defined. By (7.4.24) and (7.4.25), we have

$$\begin{aligned} \lim_{n\to\infty} H(g\varkappa_n, gy_n) &= \lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = g\varkappa, \\ \lim_{n\to\infty} H(gy_n, g\varkappa_n) &= \lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gy. \end{aligned} \tag{7.4.26}$$

Also, since F and g are continuous, so H is also continuous, then by (7.4.26) we get

$$H(g\varkappa, gy) = g\varkappa \text{ and } H(gy, g\varkappa) = gy. \tag{7.4.27}$$

Now, using (7.4.27) and the definition of H, we can obtain that $F(\varkappa, y) = g\varkappa$ and $F(y, \varkappa) = gy$.

Suppose that condition (b) holds.

Then using (7.4.24), we have $g\varkappa_n \asymp g\varkappa$ and $gy_n \asymp gy$ for sufficiently large n.

For such large n, using the triangle inequality and the monotone property of ψ, we have

$$\psi\big(d(F(\varkappa, y), g\varkappa)\big) \leq \psi\big(d(F(\varkappa, y), F(\varkappa_n, y_n)) + d(F(\varkappa_n, y_n), g\varkappa)\big).$$

Then, on taking $n \to \infty$, using the continuity of ψ and (7.4.24), we get

$$\begin{aligned} \psi\big(d(F(\varkappa, y), g\varkappa)\big) &\leq \psi\left(\lim_{n\to\infty}\big(d(F(\varkappa, y), F(\varkappa_n, y_n)) + d(F(\varkappa_n, y_n), g\varkappa)\big)\right) \\ &= \psi\left(\lim_{n\to\infty}\big(d(F(\varkappa, y), F(\varkappa_n, y_n))\big)\right) \\ &= \lim_{n\to\infty} \psi\big(d(F(\varkappa, y), F(\varkappa_n, y_n))\big). \end{aligned} \tag{7.4.28}$$

Also, by (7.4.3), we have

$$\begin{aligned} \psi\big(d(F(\varkappa, y), F(\varkappa_n, y_n))\big) \leq{}& \phi(\max\{d(g\varkappa, g\varkappa_n), d(gy, gy_n)\}) \\ &+ Ł \min\left\{\begin{matrix} d(F(\varkappa, y), g\varkappa_n), d(F(\varkappa_n, y_n), g\varkappa), \\ d(F(\varkappa, y), g\varkappa), d(F(\varkappa_n, y_n), g\varkappa_n) \end{matrix}\right\}. \end{aligned} \tag{7.4.29}$$

Using (7.4.28), (7.4.29), the properties of ϕ and Lemma 7.2.1, we can obtain

$$\psi\big(\mathrm{d}(\mathrm{F}(\varkappa,y),g\varkappa)\big) \le \phi\left(\lim_{n\to\infty}\max\{\mathrm{d}(g\varkappa,g\varkappa_n),\mathrm{d}(gy,gy_n)\}\right) + \lim_{n\to\infty} Ł\min\left\{\begin{matrix}\mathrm{d}(\mathrm{F}(\varkappa,y),g\varkappa_n),\mathrm{d}(\mathrm{F}(\varkappa_n,y_n),g\varkappa),\\ \mathrm{d}(\mathrm{F}(\varkappa,y),g\varkappa),\mathrm{d}(\mathrm{F}(\varkappa_n,y_n),g\varkappa_n)\end{matrix}\right\}$$

$$= \phi(\max\{0,0\}) + 0 = 0.$$

Hence, we obtain $\mathrm{d}(\mathrm{F}(\varkappa, y), g\varkappa) \le 0$, which implies that $\mathrm{F}(\varkappa, y) = g\varkappa$. Similarly, we can obtain that $\mathrm{F}(y, \varkappa) = gy$. Therefore, F and g have a coupled coincidence point.

Remark 7.4.1. Condition (7.4.1) provides a replacement for the MMP which has been enjoyed by various authors in their coupled fixed point results. The condition (7.4.1) is trivially satisfied if the order $\preccurlyeq$ on X is total. Also, the mappings F and g in Theorem 7.4.1 are neither commuting nor compatible and the completeness of g(X) replaces the completeness of the space X.

Next, we give an example in support of Theorem 7.4.1 as follows:

Example 7.4.1. Consider the POMS (X, $\preccurlyeq$, d) where X = (-1, 1], the natural ordering $\le$ of the real numbers as the partial ordering $\preccurlyeq$ and $\mathrm{d}(\varkappa, y) = |\varkappa - y|$ for all $\varkappa, y \in$ X. Let g: X → X and F: X × X → X be defined respectively by $g\varkappa = \frac{\varkappa^2+1}{2}$ and $\mathrm{F}(\varkappa, y) = \frac{\varkappa^2+y^2+4}{8}$. Clearly, F and g are not compatible. For $y_1 = \frac{-1}{4}$ and $y_2 = \frac{-1}{2}$, we have $gy_1 = g\left(\frac{-1}{4}\right) = \frac{17}{32} \le \frac{5}{8} = g\left(\frac{-1}{2}\right) = gy_2$ but for $\varkappa = 0$, we have $\mathrm{F}(\varkappa, y_1) = \mathrm{F}\left(0,\frac{-1}{4}\right) = \frac{65}{128} \le \frac{17}{32} = \mathrm{F}\left(0,\frac{-1}{2}\right) = \mathrm{F}(\varkappa, y_2)$, so that F does not satisfy the MgMP. Clearly, $g(X) = \left[\frac{1}{2}, 1\right]$ is complete and $\mathrm{F}(X \times X) \subseteq g(X)$. Let the mappings $\psi, \phi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined respectively by $\psi(\mathrm{f}) = \frac{\mathrm{f}}{2}$ and $\phi(\mathrm{f}) = \frac{\mathrm{f}}{4}$ for $\mathrm{f} \in \mathbb{R}^+$. Then, ψ is an ADF and ϕ is continuous such that $\psi(\mathrm{f}) > \phi(\mathrm{f})$ for all f > 0. Next, we verify the inequality (7.4.3). Let $Ł \ge 0$.

For $\varkappa, y, u, v \in$ X satisfying $g\varkappa \asymp gu$ and $gy \asymp gv$, we have

$$\psi\left(\mathrm{d}\big(\mathrm{F}(\varkappa,y),\mathrm{F}(u,v)\big)\right) = \frac{1}{2}\left(\left|\frac{\varkappa^2+y^2+4}{8} - \frac{u^2+v^2+4}{8}\right|\right) = \frac{1}{8}\left(\left|\frac{\varkappa^2+y^2}{2} - \frac{u^2+v^2}{2}\right|\right)$$

$$\le \frac{1}{8}\left(\left|\frac{\varkappa^2-u^2}{2}\right| + \left|\frac{y^2-v^2}{2}\right|\right) = \frac{1}{8}\left(\left|\frac{\varkappa^2+1}{2} - \frac{u^2+1}{2}\right| + \left|\frac{y^2+1}{2} - \frac{v^2+1}{2}\right|\right)$$

$$= \frac{1}{8}\big(\mathrm{d}(g\varkappa,gu),\mathrm{d}(gy,gv)\big) \le \frac{1}{4}(\max\{\mathrm{d}(g\varkappa,gu),\mathrm{d}(gy,gv)\})$$

$$\le \phi(\max\{\mathrm{d}(g\varkappa,gu),\mathrm{d}(gy,gv)\}) + Ł\min\left\{\begin{matrix}\mathrm{d}(\mathrm{F}(\varkappa,y),gu),\mathrm{d}(\mathrm{F}(u,v),g\varkappa),\\ \mathrm{d}(\mathrm{F}(\varkappa,y),g\varkappa),\mathrm{d}(\mathrm{F}(u,v),gu)\end{matrix}\right\}.$$

Further, the other conditions of Theorem 7.4.1 are also satisfied. Now, on applying Theorem 7.4.1, we can obtain (0, 0) as the coupled coincidence point of F and g.

Corollary 7.4.1. Let (Ӽ, ≼, ꝱ) be a POCMS and Ғ: Ӽ × Ӽ → Ӽ be the mapping satisfying (7.4.2). Suppose there exists some Ł ≥ 0 such that

$$\psi\left(ꝱ\big(Ғ(\varkappa, y), Ғ(u, v)\big)\right) \leq \phi(\max\{ꝱ(\varkappa, u), ꝱ(y, v)\}) + Ł \min\left\{\begin{matrix} ꝱ(Ғ(\varkappa, y), u), ꝱ(Ғ(u, v), \varkappa), \\ ꝱ(Ғ(\varkappa, y), \varkappa), ꝱ(Ғ(u, v), u) \end{matrix}\right\}, \quad (7.4.30)$$

for all $\varkappa$, y, u, v ∈ Ӽ with $\varkappa \asymp$ u and y ≍ v, where ψ is an ADF and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that ψ(ẛ) > ϕ(ẛ) for all ẛ > 0. Also, suppose either

(a) Ғ is continuous, or (b) Ӽ assumes Assumption 7.4.1.

Suppose that Ӽ has the property:

(P10) "there exist $\varkappa_0, y_0$ ∈ Ӽ such that $\varkappa_0 \asymp Ғ(\varkappa_0, y_0)$ and $y_0 \asymp Ғ(y_0, \varkappa_0)$".

Then, Ғ has a coupled fixed point in Ӽ.

Proof. In Theorem 7.4.1 taking g to be the identity mapping on Ӽ, the result follows immediately.

Remark 7.4.2. Corollary 7.4.1 improves the results due to Harjani et al. [58], that is Theorem 2.1.15. Setting Ł = 0 and substituting $\psi(\varkappa) - \phi(\varkappa)$ for $\phi(\varkappa)$ in Corollary 7.4.1, the condition (7.4.30) becomes

$$\psi\left(ꝱ\big(Ғ(\varkappa, y), Ғ(u, v)\big)\right) \leq \psi(\max\{ꝱ(\varkappa, u), ꝱ(y, v)\}) - \phi(\max\{ꝱ(\varkappa, u), ꝱ(y, v)\}), \quad (7.4.31)$$

which is the same condition as considered by Harjani et al. [58] (condition (7.1.7)). But then, the result obtained from Corollary 7.4.1 will be more general than the work of Harjani et al. [58] (Theorem 2.1.15) since in our results, we do not require the mapping Ғ to satisfy the MMP. The following example illustrates this fact:

Example 7.4.2. Let us consider the POCMS (Ӽ, ≼, ꝱ) with Ӽ = [-1, 1], the natural ordering ≤ of the real numbers as the partial ordering ≼ and ꝱ($\varkappa$, y) = $|\varkappa - y|$ for all $\varkappa$, y ∈ Ӽ. Let the mapping Ғ: Ӽ × Ӽ → Ӽ be defined by Ғ($\varkappa$, y) = $\frac{\varkappa^2+y^2}{8}$. Consider y_1 = 1 and $y_2 = \frac{1}{2}$, then we have $y_1 > y_2$ but for $\varkappa$ = 0, we have Ғ($\varkappa$, y_1) = Ғ(0, 1) = $\frac{1}{8} > \frac{1}{32}$ = Ғ(0, $\frac{1}{2}$) = Ғ($\varkappa$, y_2). Clearly, Ғ does not satisfy the MMP. Therefore, Theorem 2.1.15 is not applicable here. Define the mappings ψ, ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ by ψ(ẛ) = $\frac{ẛ}{2}$ and ϕ(ẛ) = $\frac{ẛ}{4}$ for ẛ ∈ $\mathbb{R}^+$. We now verify the inequality (7.4.31). For $\varkappa$, y, u, v ∈ Ӽ satisfying $\varkappa \asymp$ u and y ≍ v, we have

$$\psi\left(ꝱ\big(Ғ(\varkappa, y), Ғ(u, v)\big)\right) = \frac{1}{2}\left(\left|\frac{\varkappa^2+y^2}{8} - \frac{u^2+v^2}{8}\right|\right) = \frac{1}{16}\left(|(\varkappa^2 - u^2) - (y^2 - v^2)|\right)$$

$\leq \frac{1}{16}(|\varkappa^2 - u^2| + |y^2 - v^2|) = \frac{1}{16}(|\varkappa - u||\varkappa + u| + |y - v||y + v|)$

$\leq \frac{1}{16}\big(|\varkappa - u|(|\varkappa| + |u|) + |y - v|(|y| + |v|)\big)$

$\leq \frac{1}{16}\big(|\varkappa - u|(1 + 1) + |y - v|(1 + 1)\big)$ (since $\varkappa$, y, u, v ∈ Ꭓ)

$= \frac{1}{8}(|\varkappa - u| + |y - v|)$

$= \frac{1}{8}\big(d(\varkappa, u) + d(y, v)\big)$

$\leq \frac{1}{4}(\max\{d(\varkappa, u), d(y, v)\})$

$= \psi(\max\{d(\varkappa, u), d(y, v)\}) - \phi(\max\{d(\varkappa, u), d(y, v)\})$.

Hence, the inequality (7.4.31) holds. Since, the inequality (7.4.31) is contained in the inequality (7.4.30), on applying Corollary 7.4.1 with Remark 7.4.2, we can obtain (0, 0) as the coupled fixed point of Ꞙ.

Remark 7.4.3. (i) For $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$, we have

$$\alpha\, d(\varkappa, u) + \beta\, d(y, v) \leq (\alpha + \beta) \max\{d(\varkappa, u), d(y, v)\},$$

so that the condition (7.1.12) (which is actually due to Luong and Thuan [69]):

$$d(F(\varkappa, y), F(u, v)) \leq \alpha\, d(\varkappa, u) + \beta\, d(y, v) + L \min\begin{Bmatrix} d(F(\varkappa, y), u), d(F(u, v), \varkappa), \\ d(F(\varkappa, y), \varkappa), d(F(u, v), u) \end{Bmatrix}$$

is contained in the condition

$$d(F(\varkappa, y), F(u, v)) \leq (\alpha + \beta) \max\{d(\varkappa, u), d(y, v)\} + L \min\begin{Bmatrix} d(F(\varkappa, y), u), d(F(u, v), \varkappa), \\ d(F(\varkappa, y), \varkappa), d(F(u, v), u) \end{Bmatrix},$$

which is the condition (7.4.30) for $\psi(t) = t$ and $\phi(t) = (\alpha + \beta)\, t$, for $t \geq 0$.

Therefore, Corollary 7.4.1 is more general than the result of Luong and Thuan [69] (that is, Theorem 2.1.21). It is interesting to note that in Corollary 7.4.1 we do not require the mapping Ꞙ to satisfy MMP, whereas Theorem 2.1.21 requires this condition.

(ii) On taking $\psi(t) = t$ for all $t \geq 0$ in Theorem 7.4.1, the condition (7.4.3) becomes (7.1.13) which is due to Karapinar et al. [57]. Now, in view of Remark 7.4.1, the Theorem 7.4.1 is more general than the results of Karapinar et al. [57] (that is, Theorem 2.1.22 with the Assumption 2.1.7).

(iii) Since $\frac{d(\varkappa, u) + d(y, v)}{2} \leq \max\{d(\varkappa, u), d(y, v)\}$, so that the condition (7.1.6) (due to Bhaskar and Lakshmikantham [55]):

$$d(F(\varkappa, y), F(u, v)) \leq \frac{k}{2}[d(\varkappa, u) + d(y, v)],$$

is contained in the condition

$$\mathrm{d}(F(\varkappa, y), F(u, v)) \leq k \max\{\mathrm{d}(\varkappa, u), \mathrm{d}(y, v)\},$$

which is actually the condition (7.4.30)

$$\psi(\mathrm{d}(F(\varkappa, y), F(u, v))) \leq \phi(\max\{\mathrm{d}(\varkappa, u), \mathrm{d}(y, v)\}) + Ł \min\left\{\begin{matrix}\mathrm{d}(F(\varkappa, y), u), \mathrm{d}(F(u, v), \varkappa), \\ \mathrm{d}(F(\varkappa, y), \varkappa), \mathrm{d}(F(u, v), u)\end{matrix}\right\}$$

for $\psi(t) = t$, $\phi(t) = kt$ where $k \in (0, 1)$ for $t \geq 0$ and $Ł = 0$.

Therefore, Corollary 7.4.1 is more general than the result of Bhaskar and Lakshmikantham [55] (Theorem 2.1.14). Also, in Corollary 7.4.1, the mapping F does not satisfy the MMP which is required in Theorem 2.1.14.

(iv) Theorem 7.4.1 improves the result of Choudhury et al. [56] (that is, Theorem 2.1.18). Considering $\psi(t) - \phi(t)$ for $\phi(t)$ and $Ł = 0$ in Theorem 7.4.1, the condition (7.4.3) becomes

$$\psi(\mathrm{d}(F(\varkappa, y), F(u, v))) \leq \psi(\max\{\mathrm{d}(g\varkappa, gu), \mathrm{d}(gy, gv)\}) - \phi(\max\{\mathrm{d}(g\varkappa, gu), \mathrm{d}(gy, gv)\}),$$

which is actually the condition due to Choudhury et al. [56] (condition (7.1.10)). But in view of the Remark 7.4.1, our result is more general than the result of Choudhury et al. [56].

(v) Corollary 7.4.1 improves the result of Rasouli and Bahrampour [70] (Theorem 2.1.23). Taking $Ł = 0$, $\psi(t) = t$ and $\phi(t) = \beta(t)\, t$, for $t \geq 0$ where $\beta \in \mathfrak{R}$ in Corollary 7.4.1, the condition (7.4.30) becomes

$$\mathrm{d}(F(\varkappa, y), F(u, v)) \leq \beta(\max\{\mathrm{d}(\varkappa, u), \mathrm{d}(y, v)\}) \max\{\mathrm{d}(\varkappa, u), \mathrm{d}(y, v)\},$$

which is the condition due to Rasouli and Bahrampour [70] (that is, the condition (7.1.9)). Again, in Corollary 7.4.1, the mapping F does not satisfy the MMP, whereas Theorem 2.1.23 requires this property.

Recently, Chandok and Tas [174] established the following important result:

Theorem 7.4.2 ([174]). Let $(X, \preccurlyeq, \mathrm{d})$ be a POCMS. Suppose that $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be mappings such that g is continuous, $g(X)$ is closed, $F(X \times X) \subseteq g(X)$, the pair (F, g) is compatible, g and F satisfy the condition (7.4.1) and $Ł \geq 0$ such that

$$\mathrm{d}(F(\varkappa, y), F(u, v)) \leq \phi(\max\{\mathrm{d}(g\varkappa, gu), \mathrm{d}(gy, gv)\}) + Ł \min\left\{\begin{matrix}\mathrm{d}(F(\varkappa, y), gu), \mathrm{d}(F(u, v), g\varkappa), \\ \mathrm{d}(F(\varkappa, y), g\varkappa), \mathrm{d}(F(u, v), gu)\end{matrix}\right\} \quad (7.4.32)$$

for all $\varkappa$, y, u, v ∈ $\mathfrak{X}$ with $g\varkappa \asymp gu$ and $gy \asymp gv$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with the condition that $\phi(\mathfrak{t}) < \mathfrak{t}$ for all $\mathfrak{t} > 0$ and $\phi(\mathfrak{t}) = 0$ iff $\mathfrak{t} = 0$. Also, suppose either

(a) Ғ is continuous, or (b) $\mathfrak{X}$ assumes Assumption 7.4.1.

Suppose that $\mathfrak{X}$ has the property (P9). Then, Ғ and g have a coupled coincidence point in $\mathfrak{X}$.

Remark 7.4.4. Clearly, Theorem 7.4.1 generalizes Theorem 7.4.2.

Common Coupled Fixed Points

Now, we show the existence and uniqueness of coupled fixed points. Before we proceed, we need to consider the following:

We say that (u, v) and ($\varkappa$, y) are comparable if either

$$(u, v) \preccurlyeq (\varkappa, y) \quad \text{or} \quad (\varkappa, y) \preccurlyeq (u, v) \qquad (7.4.33)$$

and now, we will also denote this fact by $(u, v) \asymp (\varkappa, y)$.

Theorem 7.4.3. In addition to the hypotheses of Theorem 7.4.1, suppose for every $(\varkappa, y), (\varkappa^*, y^*) \in \mathfrak{X} \times \mathfrak{X}$ there exists some $(u, v) \in \mathfrak{X} \times \mathfrak{X}$ such that $(Ғ(u, v), Ғ(v, u)) \asymp (Ғ(\varkappa, y), Ғ(y, \varkappa))$ and $(Ғ(u, v), Ғ(v, u)) \asymp (Ғ(\varkappa^*, y^*), Ғ(y^*, \varkappa^*))$. If the pair of the mappings (Ғ, g) is w^* - compatible, then Ғ and g have a unique common fixed point.

Proof. By Theorem 7.4.1, the set of coupled coincidences of Ғ and g is non-empty. In order to prove the result, we first show that if ($\varkappa$, y) and ($\varkappa^*$, y^*) are coupled coincidence points, then

$$g\varkappa = g\varkappa^* \text{ and } gy = gy^*. \qquad (7.4.34)$$

By assumption, there exists some $(u, v) \in \mathfrak{X} \times \mathfrak{X}$ such that $(Ғ(u, v), Ғ(v, u)) \asymp (Ғ(\varkappa, y), Ғ(y, \varkappa))$ and $(Ғ(u, v), Ғ(v, u)) \asymp (Ғ(\varkappa^*, y^*), Ғ(y^*, \varkappa^*))$. Take $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in \mathfrak{X}$ so that $gu_1 = Ғ(u_0, v_0)$, $gv_1 = Ғ(v_0, u_0)$.

Then, as in the proof of Theorem 7.4.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = Ғ(u_n, v_n)$ and $gv_{n+1} = Ғ(v_n, u_n)$.

Further, set $\varkappa_0 = \varkappa$, $y_0 = y$, $\varkappa_0^* = \varkappa^*$, $y_0^* = y^*$ and on the same way define the sequences $\{g\varkappa_n\}$, $\{gy_n\}$ and $\{g\varkappa_n^*\}$, $\{gy_n^*\}$. Then, it is easy to obtain that

$$g\varkappa_{n+1} = Ғ(\varkappa_n, y_n),\ gy_{n+1} = Ғ(y_n, \varkappa_n)$$

and

$$g\varkappa_{n+1}^* = Ғ(\varkappa_n^*, y_n^*),\ gy_{n+1}^* = Ғ(y_n^*, \varkappa_n^*) \text{ for all } n \geq 0.$$

Since $(Ғ(u, v), Ғ(v, u)) = (gu_1, gv_1) \asymp (g\varkappa, gy) = (Ғ(\varkappa, y), Ғ(y, \varkappa)) = (g\varkappa_1, gy_1)$ are comparable, then $gu_1 \asymp g\varkappa$ and $gv_1 \asymp gy$. Then, it is easy to obtain that (gu_n, gv_n)

and $(g\varkappa, gy)$ are comparable, so that $gu_n \asymp g\varkappa$ and $gv_n \asymp gy$ for $n \in \mathbb{N}$. Now, by (7.4.3), we have

$$\psi\big(d(gu_{n+1}, g\varkappa)\big) = \psi\Big(d\big(F(u_n, v_n), F(\varkappa, y)\big)\Big)$$
$$\leq \phi(\max\{d(gu_n, g\varkappa), d(gv_n, gy)\})$$
$$+ Ł \min\left\{\begin{matrix} d(F(u_n, v_n), g\varkappa), d(F(\varkappa, y), gu_n), \\ d(F(u_n, v_n), gu_n), d(F(\varkappa, y), g\varkappa) \end{matrix}\right\},$$

which implies

$$\psi\big(d(gu_{n+1}, g\varkappa)\big) \leq \phi(\max\{d(gu_n, g\varkappa), d(gv_n, gy)\}). \quad (7.4.35)$$

Similarly, we have

$$\psi\big(d(gv_{n+1}, gy)\big) \leq \phi(\max\{d(gv_n, gy), d(gu_n, g\varkappa)\}). \quad (7.4.36)$$

Now, $\max\{d(gu_{n+1}, g\varkappa), d(gv_{n+1}, gy)\}$ is either $d(gu_{n+1}, g\varkappa)$ or $d(gv_{n+1}, gy)$, then in both the cases, using (7.4.35) and (7.4.36), we have

$$\psi(\max\{d(gu_{n+1}, g\varkappa), d(gv_{n+1}, gy)\}) \leq \phi(\max\{d(gu_n, g\varkappa), d(gv_n, gy)\}). \quad (7.4.37)$$

Denote $\sigma_n = \max\{d(gu_{n+1}, g\varkappa), d(gv_{n+1}, gy)\}$, then by (7.4.37) we have $\psi(\sigma_n) \leq \phi(\sigma_{n-1})$. Using the conditions on ψ and ϕ, we obtain that

$$\psi(\sigma_n) \leq \phi(\sigma_{n-1}) < \psi(\sigma_{n-1}).$$

Since ψ is a non-decreasing function, it follows that $\{\sigma_n\}$ is a decreasing sequence of non-negative terms, so, there exists some $\sigma \geq 0$ such that $\lim_{n\to\infty} \sigma_n = \sigma$.

We assert that $\sigma = 0$. Suppose $\sigma > 0$. Taking the limit as $n \to \infty$ in (7.4.37) and using the properties of ψ and ϕ, we can obtain $\psi(\sigma) \leq \phi(\sigma) < \psi(\sigma)$, a contradiction. Therefore, $\sigma = 0$, so that

$$\lim_{n\to\infty} \max\{d(gu_{n+1}, g\varkappa), d(gv_{n+1}, gy)\} = 0,$$

hence,

$$\lim_{n\to\infty} d(gu_{n+1}, g\varkappa) = 0 = \lim_{n\to\infty} d(gv_{n+1}, gy). \quad (7.4.38)$$

Similarly, we have

$$\lim_{n\to\infty} d(gu_{n+1}, g\varkappa^*) = 0 = \lim_{n\to\infty} d(gv_{n+1}, gy^*). \quad (7.4.39)$$

By the uniqueness of limit, we can obtain $g\varkappa = g\varkappa^*$ and $gy = gy^*$. Therefore, (7.4.34) is proved. Hence, $(g\varkappa, gy)$ is the unique point of coupled coincidence of F and g.

Also, if $(g\varkappa, gy)$ is a point of coupled coincidence of F and g, then so is $(gy, g\varkappa)$. Then, $g\varkappa = gy$ and therefore, $(g\varkappa, g\varkappa)$ is the unique point of coupled coincidence of F and g.

Next, we show that Ғ and g have a coupled common fixed point. Denote $\hat{\varkappa}$ = g$\varkappa$. Then, we have $\hat{\varkappa}$ = g$\varkappa$ = Ғ($\varkappa$, $\varkappa$). Since Ғ and g are w^*- compatible, we have

$$g\hat{\varkappa} = gg\varkappa = gҒ(\varkappa, \varkappa) = Ғ(g\varkappa, g\varkappa) = Ғ(\hat{\varkappa}, \hat{\varkappa}).$$

Therefore, (g$\hat{\varkappa}$, g$\hat{\varkappa}$) is a point of coupled coincidence of Ғ and g. By uniqueness of point of coupled coincidence of Ғ and g, we can obtain that g$\hat{\varkappa}$ = g$\varkappa$.

Therefore, $\hat{\varkappa}$ = g$\hat{\varkappa}$ = Ғ($\hat{\varkappa}$, $\hat{\varkappa}$), so that $\hat{\varkappa} \in$ Ӽ is a common fixed point of Ғ and g.

Finally, we show the uniqueness of the common fixed point of Ғ and g.

Let $\hat{y} \in$ Ӽ be any common fixed point of Ғ and g, so that we have

$$\hat{y} = g\hat{y} = Ғ(\hat{y}, \hat{y}).$$

Then (g$\hat{\varkappa}$, g$\hat{\varkappa}$) and (g$\hat{y}$, g$\hat{y}$) are two points of coupled coincidence of Ғ and g. Now, as obtained previously, we can get g$\hat{\varkappa}$ = g$\hat{y}$ and so $\hat{\varkappa}$ = g$\hat{\varkappa}$ = g$\hat{y}$ = $\hat{y}$. Hence, we have obtained the required result.

Remark 7.4.5. In addition to the hypotheses of Corollary 7.4.1, suppose for every ($\varkappa$, y), ($\varkappa^*$, y^*) ∈ Ӽ × Ӽ there exists some (u, v) ∈ Ӽ × Ӽ such that (Ғ(u, v), Ғ(v, u)) ≍ (Ғ($\varkappa$, y), Ғ(y, $\varkappa$)) and (Ғ(u, v), Ғ(v, u)) ≍ (Ғ($\varkappa^*$, y^*), Ғ(y^*, $\varkappa^*$)), then Ғ has a unique fixed point.

FRAMEWORK OF CHAPTER - VIII

In this chapter, we discuss some results for w-compatible (weakly compatible) mappings, variants of weakly commuting and compatible mappings in the context of coupled fixed point theory. This chapter deals with results in FM-spaces with some corresponding results in metric spaces. Further, the notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property are also extended for coupled fixed point problems in metric and FM-spaces.

PUBLISHED WORK:

(1) Journal of Applied Mathematics, vol. 2012, Article ID 961210, 13 pages, 2012.

WORK DONE ACCEPTED IN:

(1) Boletim da Sociedade Paranaense de Matemática, (In Press).

(2) TWMS Journal of Applied and Engineering Mathematics.

CHAPTER – VIII
COUPLED FIXED POINTS IN FM-SPACES

In this chapter, we discuss some results for w-compatible (weakly compatible) mappings, variants of weakly commuting and compatible mappings, mappings with property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property in coupled fixed point theory. This chapter consists of five sections. Section 8.1 gives a brief introduction of coupled fixed point theory in FM-spaces. In section 8.2, we discuss variants of weakly commuting and compatible mappings in coupled fixed point theory. Section 8.3 consists of coupled fixed point results for weakly compatible mappings, variants of weakly commuting and compatible mappings in FM-spaces. In section 8.4, we study the notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property and utilize these notions to generalize some existing results in context of coupled fixed point theory in FM-spaces. Section 8.5 is the application part which consists of the metrical version of some of the results proved in FM-spaces in the earlier sections of this chapter.

Present chapter deals with the results in GVFMS and we use the term **FM-space** to represent a GVFMS.

Author's Original Contributions In This Chapter Are:

Theorems: 8.3.1, 8.3.2, 8.3.3, 8.4.1, 8.4.2, 8.4.3, 8.4.4, 8.5.1, 8.5.2, 8.5.3.

Lemma: 8.2.1, 8.2.2, 8.2.3, 8.2.4, 8.2.5, 8.2.6, 8.2.7, 8.2.8, 8.3.1. 8.3.2.

Definitions: 8.2.3, 8.2.4, 8.4.1, 8.4.2, 8.4.3, 8.4.4.

Corollaries: 8.4.1.

Examples: 8.2.1, 8.2.2, 8.2.3, 8.2.4, 8.2.5, 8.2.6, 8.2.7, 8.2.8.

Remarks: 8.2.1, 8.2.2, 8.2.3, 8.2.4, 8.2.5, 8.2.6, 8.2.7, 8.4.1, 8.4.2, 8.4.3, 8.4.4, 8.4.5, 8.4.6.

8.1. INTRODUCTION

Recently, Choudhury and Kundu [60] extended the notion of compatible mappings (see, Definition 2.1.10) in coupled fixed point theory. The fuzzy counterpart of this notion was given by Hu [146] as follows:

Definition 8.1.1 ([146]). Let $(X, M, *)$ be a FM-space and $F: X \times X \to X$, $g: X \to X$ be two mappings. Then, the pair (F, g) of mappings is said to be **compatible** if

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(gx_n, gy_n), t) = 1$$

and $\lim_{n\to\infty} M(gF(y_n, \varkappa_n), F(gy_n, g\varkappa_n), t) = 1,$

for all $t > 0$, whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(\varkappa_n, y_n)$ $= \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, $y \in X$.

Utilizing the notion of compatible mappings, Hu [146] proved a common fixed point result for a ϕ - contraction in FM-spaces, where $\phi \in \Phi_\phi$ (see, **Definition 2.5.5**). Note that if $\phi \in \Phi_\phi$, then ϕ satisfies (ϕ-1), (ϕ-2)and (ϕ-3) (see, **Definition 2.5.5**). It was asserted in [146] that, "if $\phi \in \Phi_\phi$, then $\phi(t) < t$ for all $t > 0$".

Theorem 8.1.1 ([146]). Let $(X, M, *)$ be a complete FM-space with (FM-6), $*$ being a continuous t-norm of H-type. Let $F: X \times X \to X$, $g: X \to X$ be two mappings and there exists $\phi \in \Phi_\phi$ such that for all $\varkappa$, y, u, v in X and $t > 0$,

$$M(F(\varkappa, y), F(u, v), \phi(t)) \geq M(g\varkappa, gu, t) * M(gy, gv, t). \quad (8.1.1)$$

Also, suppose that $F(X \times X) \subseteq g(X)$, g is continuous and (F, g) is a pair of compatible mappings. Then, there exists a unique point u in X such that $F(u, u) = u = gu$.

On the other hand, Jain et al. [63] introduced the notion of weakly commuting mappings and their variants in coupled fixed point theory of FM-spaces. Dalal and Masmali [148] studied the notion of variants of compatible mappings in coupled fixed point theory of FM-spaces. Abbas et al. [61] defined the notion of w-compatible mappings. Later on, an equivalent notion of weakly compatible mappings came into existence.

Definition 8.1.2. The mappings $F: X \times X \to X$ and $g: X \to X$ are said to be

(i) ([61]). w-compatible,

"if $gF(\varkappa, y) = F(g\varkappa, gy)$, whenever $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$".

In this case, we say that the pair (F, g) is **w-compatible**.

(ii) ([63]). weakly compatible,

"if $gF(\varkappa, y) = F(g\varkappa, gy)$ and $gF(y, \varkappa) = F(gy, g\varkappa)$, whenever $g\varkappa = F(\varkappa, y)$ and $gy = F(y, \varkappa)$".

In this case, we say that the pair (F, g) is **weakly compatible**.

Both the notions of w-compatible and weakly compatible mappings are equivalent and we consider them as same.

Using weakly compatible mappings, Hu et al. [147] generalized Theorem 8.1.1 by proving the following result:

Theorem 8.1.2 ([147]). Let (Ҳ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being a continuous t-norm of H-type. Let Ғ: Ҳ × Ҳ → Ҳ and ꞡ: Ҳ → Ҳ be two weakly compatible mappings and there exists $\phi \in \Phi_\phi$ satisfying (8.1.1). Suppose that Ғ(Ҳ × Ҳ) ⊆ ꞡ(Ҳ) and one of the spaces Ғ(Ҳ × Ҳ) or ꞡ(Ҳ) is complete. Then, there exists a unique point u in Ҳ such that Ғ(u, u) = u = ꞡu.

Jain et al. [63] generalized Theorem 8.1.1 for two pairs of weakly compatible mappings under the following result:

Theorem 8.1.3 ([63]). Let (Ҳ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being a continuous t-norm of H-type. Let Ⱥ, Ƀ: Ҳ × Ҳ → Ҳ and Ş, Ţ: Ҳ → Ҳ be four mappings and there exists $\phi \in \Phi_\phi$ such that for all ϰ, y, u, v in Ҳ and ƭ > 0,

$$Ϻ(Ⱥ(ϰ, y), Ƀ(u, v), \phi(ƭ)) \geq Ϻ(Şϰ, Ţu, ƭ) * Ϻ(Şy, Ţv, ƭ). \qquad (8.1.2)$$

Also, suppose that Ⱥ(Ҳ × Ҳ) ⊆ Ţ(Ҳ), Ƀ(Ҳ × Ҳ) ⊆ Ş(Ҳ), the pairs (Ⱥ, Ş) and (Ƀ, Ţ) are weakly compatible, one of the subspaces Ⱥ(Ҳ × Ҳ) or Ţ(Ҳ) and one of Ƀ(Ҳ × Ҳ) or Ş(Ҳ) are complete. Then, there exists a unique point ɑ in Ҳ such that Ⱥ(ɑ, ɑ) = Şɑ = ɑ = Ţɑ = Ƀ(ɑ, ɑ).

For convenience, in our results, we denote

$$[Ϻ(ϰ, y, ƭ)]^i = \underbrace{Ϻ(ϰ, y, ƭ) * Ϻ(ϰ, y, ƭ) * \ldots * Ϻ(ϰ, y, ƭ)}_{i}, \text{ for all } i \in \mathbb{N}.$$

8.2 VARIANTS OF WEAKLY COMMUTING AND COMPATIBLE MAPPINGS

This section deals with the variants of weakly commuting and compatible mappings in coupled fixed point theory.

Recently, Jain et al. [63] extended the variants of weakly commuting mappings from ordinary fixed point theory to coupled fixed point theory in FϺ-spaces as follows:

Definition 8.2.1 ([63]). Let (Ҳ, Ϻ, ∗) be a FϺ-space and Ғ: Ҳ × Ҳ → Ҳ, ꞡ: Ҳ → Ҳ be two mappings. Then, the pair (Ғ, ꞡ) of mappings is said to be

(i) **Weakly commuting** (we write, **WC**), if

Ϻ(Ғ(ꞡϰ, ꞡy), ꞡҒ(ϰ, y), ƭ) ≥ Ϻ(Ғ(ϰ, y), ꞡϰ, ƭ),

Ϻ(Ғ(ꞡy, ꞡϰ), ꞡҒ(y, ϰ), ƭ) ≥ Ϻ(Ғ(y, ϰ), ꞡy, ƭ) for all ϰ, y in Ҳ and ƭ > 0.

(ii) **R-weakly commuting** (we write, **R-WC**), if there exists some R > 0 such that

$$M(F(g\varkappa, gy), gF(\varkappa, y), t) \geq M(F(\varkappa, y), g\varkappa, t/R),$$

$$M(F(gy, g\varkappa), gF(y, \varkappa), t) \geq M(F(y, \varkappa), gy, t/R) \text{ for all } \varkappa, y \text{ in } X \text{ and } t > 0.$$

(iii) **R-weakly commuting of type (A_F)** (we write, **R-WC(A_F)**), if there exists some R > 0 such that

$$M(F(g\varkappa, gy), gg\varkappa, t) \geq M(F(\varkappa, y), g\varkappa, t/R),$$

$$M(F(gy, g\varkappa), ggy, t) \geq M(F(y, \varkappa), gy, t/R) \text{ for all } \varkappa, y \text{ in } X \text{ and } t > 0.$$

(iv) **R-weakly commuting of type (A_g)** (we write, **R-WC(A_g)**), if there exists some R > 0 such that

$$M(gF(\varkappa, y), F(F(\varkappa, y), F(y, \varkappa)), t) \geq M(F(\varkappa, y), g\varkappa, t/R),$$

$$M(gF(y, \varkappa), F(F(y, \varkappa), F(\varkappa, y)), t) \geq M(F(y, \varkappa), gy, t/R)$$

for all $\varkappa$, y in X and t > 0.

(v) **R-weakly commuting of type (P)** (we write, **R-WC(P)**), if there exists some R > 0 such that

$$M(F(F(\varkappa, y), F(y, \varkappa)), gg\varkappa, t) \geq M(F(\varkappa, y), g\varkappa, t/R),$$

$$M(F(F(y, \varkappa), F(\varkappa, y)), ggy, t) \geq M(F(y, \varkappa), gy, t/R)$$

for all $\varkappa$, y in X and t > 0.

Now, we discuss some illustrations for these mappings as follows:

Example 8.2.1. Let $X = \mathbb{R}^+\backslash\{0\}$. Define $a * b = ab$ for $a, b \in [0, 1]$ and $M(\varkappa, y, t) = \frac{t}{t+|\varkappa-y|}$ for all $\varkappa$, y in X and t > 0. Then, (X, M, ∗) is a FM-space. Also, define $F: X \times X \to X$ and $g: X \to X$ respectively by $F(\varkappa, y) = \frac{\varkappa+y}{2}$ and $g\varkappa = \frac{\varkappa}{2}$ for $\varkappa$, y in X. Now, for all $\varkappa$, y in X and t > 0, we have

$$M(F(g\varkappa, gy), gF(\varkappa, y), t) = 1 > \frac{2t}{2t+y} = M(F(\varkappa, y), g\varkappa, t)$$

and $$M(F(gy, g\varkappa), gF(y, \varkappa), t) = 1 > \frac{2t}{2t+\varkappa} = M(F(y, \varkappa), gy, t),$$

so that the pair (F, g) is WC.

Moreover, for all $\varkappa$, y in X and t > 0, we have

$$M(F(g\varkappa, gy), gF(\varkappa, y), t) = 1 > \frac{2t}{2t+Ry} = M(F(\varkappa, y), g\varkappa, t/R)$$

and $$M(F(gy, g\varkappa), gF(y, \varkappa), t) = 1 > \frac{2t}{2t+R\varkappa} = M(F(y, \varkappa), gy, t/R), \text{ for each } R > 0,$$

which implies that the pair (F, g) is R-WC for each R > 0.

Also, for $R \geq \frac{1}{2}$, the pair (F, g) is R-WC(A_F), since for all $\varkappa$, y in X and t > 0, we have

$$M(F(g\varkappa, gy), gg\varkappa, t) = \frac{4t}{4t+y} \geq \frac{2t}{2t+Ry} = M(F(\varkappa, y), g\varkappa, t/R)$$

and $\quad \mathcal{M}(F(gy, g\varkappa), ggy, t) = \frac{4t}{4t+\varkappa} \geq \frac{2t}{2t+R\varkappa} = \mathcal{M}(F(y, \varkappa), gy, t/R)$.

Now, the pair (F, g) is R-WC for R > 0 but R-WC(A_F) for $R \geq \frac{1}{2}$.

Remark 8.2.1. Example 8.2.1 shows that the pair of R-WC mappings need not be R-WC(A_F) for the same value of R.

Example 8.2.2. Let $X = [1, \infty)$. Define $a * b = ab$ for $a, b \in [0, 1]$ and $\mathcal{M}(\varkappa, y, t) = \frac{t}{t+|\varkappa-y|}$ for all $\varkappa$, y in X and $t > 0$. Then $(X, \mathcal{M}, *)$ is a FM-space. Also, let $F: X \times X \to X$ and $g: X \to X$ be defined by $F(\varkappa, y) = 2(\varkappa + y) + 1$ and $g\varkappa = 2\varkappa + 2$ for $\varkappa$, y in X. Now, the pair (F, g) is not commuting, since $F(g\varkappa, gy) = [4(\varkappa + y) + 9] \neq [4(\varkappa + y) + 4] = gF(\varkappa, y)$ for $\varkappa$, y in X.

Also, for $\varkappa$, y in X and $t > 0$, we have

$$\mathcal{M}(F(g\varkappa, gy), gg\varkappa, t) = \frac{t}{t+|4y+3|} \geq \frac{t}{t+R|2y-1|} = \mathcal{M}(F(\varkappa, y), g\varkappa, t/R),$$

$$\mathcal{M}(F(gy, g\varkappa), ggy, t) = \frac{t}{t+|4x+3|} \geq \frac{t}{t+R|2x-1|} = \mathcal{M}(F(y, \varkappa), gy, t/R) \text{ for } R \geq 7,$$

which shows that the pair (F, g) is R-WC(A_F) for $R \geq 7$.

Further, it is easy to see that the pair (F, g) is R-WC for each $R \geq 5$ but neither WC nor R-WC(A_g) and R-WC(P) for any $R > 0$.

Remark 8.2.2. The pair of mappings which is R-WC for some value of $R > 0$ need not be WC nor R-WC(A_F), R-WC(A_g), R-WC(P) for the same value of R.

Example 8.2.3. Let $X = [1, \infty)$. Define $a * b = ab$ for $a, b \in [0, 1]$ and $\mathcal{M}(\varkappa, y, t) = \frac{t}{t+|\varkappa-y|}$ for all $\varkappa$, y in X and $t > 0$. Then $(X, \mathcal{M}, *)$ is a FM-space. Also, let $F: X \times X \to X$ and $g: X \to X$ be defined by $F(\varkappa, y) = \frac{\varkappa}{2}$ and $g\varkappa = \varkappa^2$ for $\varkappa$, y in X. Then, the pair (F, g) is not commuting, since $F(g\varkappa, gy) = F(\varkappa^2, y^2) = \frac{\varkappa^2}{2} \neq \frac{\varkappa^2}{4} = gF(\varkappa, y)$ for $\varkappa$, y in X. Also, for all $\varkappa$, y in X and $t > 0$, we have

$$\mathcal{M}(F(g\varkappa, gy), gF(\varkappa, y), t) = \frac{t}{t+\left|\frac{\varkappa^2}{4}\right|} \geq \frac{t}{t+\left|\varkappa^2-\frac{\varkappa}{2}\right|} = \mathcal{M}(F(\varkappa, y), g\varkappa, t),$$

$$\mathcal{M}(F(gy, g\varkappa), gF(y, \varkappa), t) = \frac{t}{t+\left|\frac{y^2}{4}\right|} \geq \frac{t}{t+\left|y^2-\frac{y}{2}\right|} = \mathcal{M}(F(y, \varkappa), gy, t),$$

which shows that the pair (F, g) is WC.

Further, for all $\varkappa$, y in X and $t > 0$, we have

$$\mathcal{M}(F(g\varkappa, gy), gF(\varkappa, y), t) = \frac{t}{t+\left|\frac{\varkappa^2}{4}\right|} \geq \frac{t}{t+R\left|\varkappa^2-\frac{\varkappa}{2}\right|} = \mathcal{M}(F(\varkappa, y), g\varkappa, t/R),$$

$$\text{M}(\text{Ғ}(gy, g\varkappa), g\text{Ғ}(y, \varkappa), t) = \frac{t}{t + \left|\frac{y^2}{4}\right|} \geq \frac{t}{t + R\left|y^2 - \frac{y}{2}\right|} = \text{M}(\text{Ғ}(y, \varkappa), gy, t/R) \text{ for } R \geq \frac{1}{2},$$

which shows that the pair (Ғ, g) is R-WC for $R \geq \frac{1}{2}$.

Also, the pair (Ғ, g) is R-WC(A_g) for $R \geq \frac{1}{4}$ but neither R-WC($A_Ғ$) nor R-WC(P) for any R > 0.

Clearly, for $R = \frac{1}{4}$, the pair (Ғ, g) is R-WC(A_g) but not R-WC.

Remark 8.2.3. In general, every pair of commuting mappings is always WC but the converse need not be true. Further, the pair of R-WC(A_g) mappings need not be R-WC nor R-WC($A_Ғ$), R-WC(P).

Example 8.2.4. Let Ӽ = [1, ∞). Define $a * b = ab$ for a, b ∈ [0, 1] and $\text{M}(\varkappa, y, t) = \frac{t}{t+|\varkappa - y|}$ for all ϰ, y in Ӽ and t > 0. Then, (Ӽ, M, ∗) is a FM-space. Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be defined by $Ғ(\varkappa, y) = 2\varkappa + 1$ and $g\varkappa = \varkappa + 1$ for ϰ, y in Ӽ.

Now, for ϰ, y in Ӽ, we have

$Ғ(g\varkappa, gy) = 2\varkappa + 3$, $Ғ(gy, g\varkappa) = 2y + 3$, $gҒ(\varkappa, y) = 2\varkappa + 2$, $gҒ(y, \varkappa) = 2y + 2$, $Ғ(Ғ(\varkappa, y), Ғ(y, \varkappa)) = 4\varkappa + 3$, $Ғ(Ғ(y, \varkappa), Ғ(\varkappa, y)) = 4y + 3$, $gg\varkappa = \varkappa + 2$, $ggy = y + 2$. Then, the pair (Ғ, g) is R-WC for R ≥ 1 (and so is WC), R-WC($A_Ғ$) for R ≥ 2, R-WC(A_g) for R ≥ 3, R-WC(P) for R ≥ 4. Further, the pair (Ғ, g) is not commuting.

Recently, Dalal and Masmali [148] gave the notions of variants of compatible mappings in coupled fixed point theory in FM-spaces. We summarize these notions as follows:

Definition 8.2.2 ([148]). Let (Ӽ, M, ∗) be a FM-space and Ғ: Ӽ × Ӽ → Ӽ, g: Ӽ → Ӽ be the mappings. If whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ӽ such that $\lim_{n\to\infty} Ғ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Ғ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ӽ, then, the pair (Ғ, g) is said to be

(i) **Compatible of type (A)** (we write, **COM(A)**), if

$$\lim_{n\to\infty} \text{M}(Ғ(g\varkappa_n, gy_n), g^2\varkappa_n, t) = 1, \lim_{n\to\infty} \text{M}(Ғ(gy_n, g\varkappa_n), g^2y_n, t) = 1$$

and

$$\lim_{n\to\infty} \text{M}\big(gҒ(\varkappa_n, y_n), Ғ\big(Ғ(\varkappa_n, y_n), Ғ(y_n, \varkappa_n)\big), t\big) = 1,$$

$$\lim_{n\to\infty} \text{M}\big(gҒ(y_n, \varkappa_n), Ғ\big(Ғ(y_n, \varkappa_n), Ғ(\varkappa_n, y_n)\big), t\big) = 1, \quad \text{for } t > 0;$$

(ii) **Compatible of type (B)** (we write, **COM(B)**), if

$$\lim_{n\to\infty} M\,(F(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq \frac{1}{2}\begin{cases} \lim_{n\to\infty} M(F(g\varkappa_n, gy_n), F(\varkappa, y), t) \\ + \lim_{n\to\infty} M(F(\varkappa, y), F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), t), \end{cases}$$

$$\lim_{n\to\infty} M\,(F(gy_n, g\varkappa_n), g^2y_n, t) \geq \frac{1}{2}\begin{cases} \lim_{n\to\infty} M(F(gy_n, g\varkappa_n), F(y, \varkappa), t) \\ + \lim_{n\to\infty} M(F(y, x), F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), t) \end{cases}$$

and

$$\lim_{n\to\infty} M(gF(\varkappa_n, y_n), F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), t) \geq \frac{1}{2}\begin{cases} \lim_{n\to\infty} M(gF(\varkappa_n, y_n), g\varkappa, t) \\ + \lim_{n\to\infty} M(g\varkappa, g^2\varkappa_n, t), \end{cases}$$

$$\lim_{n\to\infty} M(gF(y_n, \varkappa_n), F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), t) \geq \frac{1}{2}\begin{cases} \lim_{n\to\infty} M(gF(y_n, \varkappa_n), gy, t) \\ + \lim_{n\to\infty} M(gy, g^2y_n, t), \end{cases}$$

for t > 0;

(iii) **Compatible of type (P)** (we write, **COM(P)**), if

$$\lim_{n\to\infty} M(F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), g^2\varkappa_n, t) = 1$$

and $$\lim_{n\to\infty} M(F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), g^2y_n, t) = 1, \quad \text{for } t > 0;$$

(iv) **Compatible of type (C)** (we write, **COM(C)**), if

$$\lim_{n\to\infty} M(F(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq \frac{1}{3}\begin{cases} \lim_{n\to\infty} M(F(g\varkappa_n, gy_n), F(\varkappa, y), t) \\ + \lim_{n\to\infty} M(F(\varkappa, y), g^2\varkappa_n, t) \\ + \lim_{n\to\infty} M(F(\varkappa, y), F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), t), \end{cases}$$

$$\lim_{n\to\infty} M(F(gy_n, g\varkappa_n), g^2y_n, t) \geq \frac{1}{3}\begin{cases} \lim_{n\to\infty} M(F(gy_n, g\varkappa_n), F(y, \varkappa), t) \\ + \lim_{n\to\infty} M(F(y, \varkappa), g^2y_n, t) \\ + \lim_{n\to\infty} M(F(y, \varkappa), F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), t) \end{cases}$$

and

$$\lim_{n\to\infty} M(gF(\varkappa_n, y_n), F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), t)$$

$$\geq \frac{1}{3}\begin{cases} \lim_{n\to\infty} M(gF(\varkappa_n, y_n), g\varkappa, t) \\ + \lim_{n\to\infty} M(g\varkappa, F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), t) \\ + \lim_{n\to\infty} M(g\varkappa, g^2\varkappa_n, t), \end{cases}$$

$$\lim_{n\to\infty} M(gF(y_n, \varkappa_n), F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), t)$$

$$\geq \frac{1}{3}\begin{cases} \lim_{n\to\infty} M(gF(y_n, \varkappa_n), gy, t) \\ + \lim_{n\to\infty} M(gy, F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), t) \\ + \lim_{n\to\infty} M(gy, g^2y_n, t), \end{cases}$$

for t > 0;

(v) **Compatible of type (A_F)** (we write, **COM(A_F)**), if

$\lim_{n\to\infty} M(F(gx_n, gy_n), ggx_n, t) = 1$, $\lim_{n\to\infty} M(F(gy_n, gx_n), ggy_n, t) = 1$, for $t > 0$;

(vi) **Compatible of type (A_g)** (we write, **COM(A_g)**), if

$\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) = 1$,

$\lim_{n\to\infty} M(gF(y_n, x_n), F(F(y_n, x_n), F(x_n, y_n)), t) = 1$, for $t > 0$.

The following example illustrates that the pair of compatible mappings need not be COM(A), COM(P), COM(A_F), COM(A_g):

Example 8.2.5. Let $X = \mathbb{R}$. Define $a * b = ab$ for $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$. Then, $(X, M, *)$ is a FM-space. Let $F: X \times X \to X$ and $g: X \to X$ be defined by

$$F(x, y) = \begin{cases} \frac{1}{(xy)^3}, & xy \neq 0 \\ 3, & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 4, & x = 0 \end{cases} \quad \text{for} \quad x, y \in X.$$

The pair (F, g) is compatible but neither COM(A) nor COM(P), COM(A_F), COM(A_g). For, let $\{x_n = n^2, n \geq 1\}$ and $\{y_n = 2n^2, n \geq 1\}$. Then

$$\lim_{n\to\infty} F(x_n, y_n) = 0 = \lim_{n\to\infty} gx_n \text{ and } \lim_{n\to\infty} F(y_n, x_n) = 0 = \lim_{n\to\infty} gy_n.$$

Also, since $F(gx_n, gy_n) = 64n^{24}$, $F(gy_n, gx_n) = 64n^{24}$, $g^2x_n = n^8$, $g^2y_n = 16n^8$, $F(F(x_n, y_n), F(y_n, x_n)) = (64n^{24})^3$, $F(F(y_n, x_n), F(x_n, y_n)) = (64n^{24})^3$, $gF(x_n, y_n) = 64n^{24}$, $gF(y_n, x_n) = 64n^{24}$, we have

$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) \neq 1$, $\lim_{n\to\infty} M(F(F(x_n, y_n), F(y_n, x_n)), g^2x_n, t) \neq 1$, $\lim_{n\to\infty} M(gF(x_n, y_n), F(F(x_n, y_n), F(y_n, x_n)), t) \neq 1$.

Thus, the pair (F, g) is neither COM(A) nor COM(P), COM(A_F), COM(A_g).

Further, for the sequences $\{x_n\}$ and $\{y_n\}$, with $\lim_{n\to\infty} F(x_n, y_n) = x = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n, x_n) = y = \lim_{n\to\infty} gy_n$ for some x, y in X. Also, $gF(x_n, y_n) = (x_n y_n)^6 = F(gx_n, gy_n)$ and $gF(y_n, x_n) = (x_n y_n)^6 = F(gy_n, gx_n)$, so that, we have

$$\lim_{n\to\infty} M(gF(x_n, y_n), F(gx_n, gy_n), t) = \lim_{n\to\infty} \frac{t}{t + |gF(x_n, y_n) - F(gx_n, gy_n)|} = 1.$$

Similarly, $\lim_{n\to\infty} M(gF(y_n, x_n), F(gy_n, gx_n), t) = 1$, so that the pair (F, g) is compatible.

The following example illustrates that if pair of mappings is COM(A), then, it may not be compatible:

Example 8.2.6. Let $\mathfrak{X}$ = [0, 6]. Define $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in [0, 1]$ and $M(\varkappa, y, \mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}+|\varkappa-y|}$ for all $\varkappa$, y in $\mathfrak{X}$ and $\mathfrak{t} > 0$. Then, ($\mathfrak{X}$, M, $*$) is a FM-space. Let $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $g: \mathfrak{X} \to \mathfrak{X}$ by

$$F(\varkappa, y) = \begin{cases} \frac{\varkappa+y}{2}, & \text{if both } \varkappa, y \in [0, 3) \\ 6, & \text{otherwise} \end{cases} \quad \text{and} \quad g\varkappa = \begin{cases} 6 - x, & \text{if } \varkappa \in [0, 3) \\ 6, & \text{otherwise} \end{cases} \text{ for } \varkappa, y \text{ in } \mathfrak{X}.$$

Let $\left\{\varkappa_n = 3 - \frac{1}{n}, n \geq 1\right\}$ and $\left\{y_n = 3 - \frac{1}{2n}, n \geq 1\right\}$ be two sequences. Then, we get

$$\lim_{n\to\infty} F(\varkappa_n, y_n) = 3 = \lim_{n\to\infty} g\varkappa_n \text{ and } \lim_{n\to\infty} F(y_n, \varkappa_n) = 3 = \lim_{n\to\infty} gy_n.$$

Now, we have $gF(\varkappa_n, y_n) = \left(3 + \frac{3}{4n}\right)$, $gF(y_n, \varkappa_n) = \left(3 + \frac{3}{4n}\right)$, $F(g\varkappa_n, gy_n) = 6$, $F(gy_n, g\varkappa_n) = 6$, $g^2\varkappa_n = 6$, $g^2 y_n = 6$, $F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big) = \left(3 - \frac{3}{4n}\right)$, $F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big) = \left(3 - \frac{3}{4n}\right)$.

Now, the pair (F, g) is not compatible, since

$$\lim_{n\to\infty} M(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n), \mathfrak{t}) = \lim_{n\to\infty} M(3 + \frac{3}{4n}, 6, \mathfrak{t}) \neq 1, \text{ for } \mathfrak{t} > 0.$$

But the pair (F, g) is COM(A), since

$$\lim_{n\to\infty} M(F(g\varkappa_n, gy_n), g^2\varkappa_n, \mathfrak{t}) = \lim_{n\to\infty} \frac{\mathfrak{t}}{\mathfrak{t}+|6-6|} = 1,$$

$$\lim_{n\to\infty} M(F(gy_n, g\varkappa_n), g^2 y_n, \mathfrak{t}) = \lim_{n\to\infty} \frac{\mathfrak{t}}{\mathfrak{t}+|6-6|} = 1$$

and

$$\lim_{n\to\infty} M\big(gF(\varkappa_n, y_n), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big), \mathfrak{t}\big) = \lim_{n\to\infty} \frac{\mathfrak{t}}{\mathfrak{t}+\left|\frac{6}{4n}\right|} = 1,$$

$$\lim_{n\to\infty} M\big(gF(y_n, \varkappa_n), F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big), \mathfrak{t}\big) = \lim_{n\to\infty} \frac{\mathfrak{t}}{\mathfrak{t}+\left|\frac{6}{4n}\right|} = 1, \text{ for } \mathfrak{t} > 0.$$

Lemma 8.2.1. Let ($\mathfrak{X}$, M, $*$) be a FM-space and $F: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g: \mathfrak{X} \to \mathfrak{X}$ be two mappings such that the pair (F, g) is COM(A) and any one of F or g is continuous, then, the pair (F, g) is compatible.

Proof. W.L.O.G., let g be continuous. Suppose that $\{\varkappa_n\}$ and $\{y_n\}$ be two sequences in $\mathfrak{X}$ such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, y in $\mathfrak{X}$. Now,

$$M(F(g\varkappa_n, gy_n), gF(\varkappa_n, y_n), \mathfrak{t})$$
$$\geq M(F(g\varkappa_n, gy_n), g^2\varkappa_n, \mathfrak{t}/2) * M(g^2\varkappa_n, gF(\varkappa_n, y_n), \mathfrak{t}/2),$$

on letting $n \to \infty$ in the above inequality, then, since the pair (F, g) is COM(A) and g is continuous, it follows that $\lim_{n\to\infty} M(F(g\varkappa_n, gy_n), gF(\varkappa_n, y_n), \mathfrak{t}) = 1$.

Similarly, we can get $\lim_{n\to\infty} M(F(gy_n, gx_n), gF(y_n, x_n), t) = 1$. Therefore, the pair (F, g) is compatible.

Likewise, it can be proved that if F is continuous and the pair (F, g) is COM(A), then, the pair (F, g) is also compatible.

Lemma 8.2.2. Let (X, M, ∗) be a FM-space and F: X × X → X, g: X → X be two mappings such that the pair (F, g) is compatible and both F, g are continuous, then, the pair (F, g) is also COM(A).

Proof. Result follows immediately by using the definitions.

Lemma 8.2.3. Let (X, M, ∗) be a FM-space and F: X × X → X, g: X → X be two mappings. If g is continuous, then the pair (F, g) is COM(A_F) iff the pair (F, g) is compatible.

Proof. Let g be continuous. Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some x, y in X.

Let the pair (F, g) be COM(A_F).

Now, $$M(F(gx_n, gy_n), gF(x_n, y_n), t)$$
$$\geq M(F(gx_n, gy_n), g^2x_n, t/2) * M(g^2x_n, gF(x_n, y_n), t/2),$$

on letting n → ∞ and by continuity of g, it follows that

$$\lim_{n\to\infty} M(F(gx_n, gy_n), gF(x_n, y_n), t) = 1.$$

Similarly, $$\lim_{n\to\infty} M(F(gy_n, gx_n), gF(y_n, x_n), t) = 1.$$

Hence, the pair (F, g) is compatible.

We conclude the proof by showing that the pair (F, g) is COM(A_F), if the pair (F, g) is compatible. For,

$$M(F(gx_n, gy_n), g^2x_n, t)$$
$$\geq M(F(gx_n, gy_n), gF(x_n, y_n), t/2) * M(gF(x_n, y_n), g^2x_n, t/2),$$

then, on letting n → ∞ and using the continuity of g, we get

$$\lim_{n\to\infty} M(F(gx_n, gy_n), g^2x_n, t) = 1.$$

Similarly, $$\lim_{n\to\infty} M(F(gy_n, gx_n), g^2y_n, t) = 1.$$

Therefore, the pair (F, g) is COM(A_F). This completes the proof.

Lemma 8.2.4. Let (X, M, ∗) be a FM-space and F: X × X → X, g: X → X be two mappings. If F is continuous, then, the pair (F, g) is COM(A_g) iff the pair (F, g) is compatible.

Proof. The proof can be obtained on similar lines of the proof of Lemma 8.2.3.

Lemma 8.2.5. Let ($\mathfrak{X}$, $\mathcal{M}$, $*$) be a FM-space and F: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, g: $\mathfrak{X} \to \mathfrak{X}$ be two mappings. If the pair (F, g) is COM(A), then it is COM(B), COM(P), COM(A_F) and COM(A_g).

Proof. Let the pair (F, g) be COM(A). Then by definitions, the pair (F, g) is also COM(B), COM(A_F) and COM(A_g). We shall show that the pair (F, g) is COM(P).

For, let $\{\varkappa_n\}$ and $\{y_n\}$ be two sequences in $\mathfrak{X}$ such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, y in $\mathfrak{X}$.

Now, for $\mathfrak{t} > 0$, we have

$\mathcal{M}(F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), g^2\varkappa_n, \mathfrak{t})$

$\geq \mathcal{M}(F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), F(g\varkappa_n, gy_n), \mathfrak{t}/2) * \mathcal{M}(F(g\varkappa_n, gy_n), g^2\varkappa_n, \mathfrak{t}/2)$.

Taking $n \to \infty$ in above inequality and using the definition of pair of COM(A), we get $\lim_{n\to\infty} \mathcal{M}(F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), g^2\varkappa_n, \mathfrak{t}) = 1$.

Similarly, we can get $\lim_{n\to\infty} \mathcal{M}(F(F(y_n, \varkappa_n), F(\varkappa_n, y_n)), g^2 y_n, \mathfrak{t}) = 1$.

Therefore, the pair (F, g) is COM(P).

Remark 8.2.4. Using Lemma 8.2.5, the Example 8.2.6 illustrates the fact that "the pair of the mappings which is COM(B)/ COM(P)/ COM(A_F)/ COM(A_g) need not be compatible".

Lemma 8.2.6. Let ($\mathfrak{X}$, $\mathcal{M}$, $*$) be a FM-space and F: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, g: $\mathfrak{X} \to \mathfrak{X}$ be two continuous mappings. Then, the pair (F, g) is COM(B)/ COM(C)/ COM(P) iff it is compatible.

Proof. Let the pair (F, g) is COM(B). We shall show that the pair (F, g) is compatible.

For, let $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in $\mathfrak{X}$ such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, y in $\mathfrak{X}$. Now, since the pair (F, g) is COM(B), then, using the continuity of F and g, by condition

$$\lim_{n\to\infty} \mathcal{M}(F(g\varkappa_n, gy_n), g^2\varkappa_n, \mathfrak{t}) \geq \frac{1}{2} \begin{cases} \lim_{n\to\infty} \mathcal{M}(F(g\varkappa_n, gy_n), F(\varkappa, y), \mathfrak{t}) \\ + \lim_{n\to\infty} \mathcal{M}(F(\varkappa, y), F(F(\varkappa_n, y_n), F(y_n, \varkappa_n)), \mathfrak{t}), \end{cases}$$

we obtain that $\mathcal{M}(F(\varkappa, y), g\varkappa, \mathfrak{t}) \geq 1$, that is, $F(\varkappa, y) = g\varkappa$. Similarly, it can be obtained that $F(y, \varkappa) = gy$. Now, for $\mathfrak{t} > 0$, we get

$$\begin{aligned} &\mathcal{M}(gF(\varkappa_n, y_n), F(g\varkappa_n, gy_n), \mathfrak{t}) \\ &\qquad \geq \mathcal{M}(gF(\varkappa_n, y_n), g\varkappa, \mathfrak{t}/2) * \mathcal{M}(g\varkappa, F(g\varkappa_n, gy_n), \mathfrak{t}/2), \end{aligned}$$

then, on taking n → ∞ in the last inequality and using the continuity of Ӻ, g in the last inequality, we get $\lim_{n\to\infty} M(gӺ(\varkappa_n, y_n), Ӻ(g\varkappa_n, gy_n), t) = 1$. Similarly, we can obtain $\lim_{n\to\infty} M(gӺ(y_n, \varkappa_n), Ӻ(gy_n, g\varkappa_n), t) = 1$. Hence the pair (Ӻ, g) is compatible.

Conversely, let the pair (Ӻ, g) be compatible.

To show that it is COM(B). For, let $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ӿ such that $\lim_{n\to\infty} Ӻ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Ӻ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ӿ.

Now, we have

$$M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, t)$$
$$\geq M(Ӻ(g\varkappa_n, gy_n), gӺ(\varkappa_n, y_n), t/2) * M(gӺ(\varkappa_n, y_n), g^2\varkappa_n, t/2),$$

then, on taking n → ∞ and using the compatible hypothesis of (Ӻ, g) with continuity of g, we get $\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq 1$, that is, $\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, t) = 1$.

Also, by continuity of Ӻ, we get

$$\frac{1}{2}\left\{\begin{array}{c}\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), Ӻ(\varkappa, y), t) \\ + \lim_{n\to\infty} M(Ӻ(\varkappa, y), Ӻ(Ӻ(\varkappa_n, y_n), Ӻ(y_n, \varkappa_n)), t)\end{array}\right\} = 1.$$

Hence, we conclude that

$$\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq \frac{1}{2}\left\{\begin{array}{c}\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), Ӻ(\varkappa, y), t) \\ + \lim_{n\to\infty} M(Ӻ(\varkappa, y), Ӻ(Ӻ(\varkappa_n, y_n), Ӻ(y_n, \varkappa_n)), t).\end{array}\right.$$

Further, all the other conditions for (Ӻ, g) to be COM(B) holds.

Likewise, it can be easily proved that if Ӻ and g are both continuous, then, the pair (Ӻ, g) is COM(C)/ COM(P) iff the pair (Ӻ, g) is compatible.

Example 8.2.7. Let Ӿ = [0, 2]. Define $a * b = ab$ for a, b ∈ [0, 1] and $M(\varkappa, y, t) = \frac{t}{t+|\varkappa - y|}$ for all ϰ, y in Ӿ and t > 0. Then, (Ӿ, M, ∗) is a FM-space. Let Ӻ: Ӿ × Ӿ → Ӿ and g: Ӿ → Ӿ be defined by

$$Ӻ(\varkappa, y) = \begin{cases} \frac{1}{2} + \varkappa, & \text{if } \varkappa, y \in [0, \frac{1}{2}) \\ 2, & \text{if } \varkappa = y = \frac{1}{2} \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad g\varkappa = \begin{cases} \frac{1}{2} - \varkappa, & \text{if } \varkappa \in [0, \frac{1}{2}) \\ \varkappa - \frac{1}{2}, & \text{if } \varkappa \in (\frac{1}{2}, 1) \\ 1, & \text{otherwise.} \end{cases}$$

Then, the pair (Ӻ, g) is COM(B) but neither compatible nor COM(A), COM(C), COM(P).

Lemma 8.2.7. Let (Ӿ, M, ∗) be a FM-space and Ӻ: Ӿ × Ӿ → Ӿ, g: Ӿ → Ӿ be two mappings. If the pair (Ӻ, g) is COM(B) (or COM(C)) and both Ӻ, g are continuous, then, the pair (Ӻ, g) is COM(A).

Proof. Let the pair (Ꞙ, g) be COM(B) and both Ꞙ, g are continuous. We show that the pair (Ꞙ, g) is COM(A).

For, let $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ҳ such that $\lim_{n\to\infty} Ꞙ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Ꞙ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ҳ. Since, the pair (Ꞙ, g) is COM(B), we have

$$\lim_{n\to\infty} M(Ꞙ(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq \frac{1}{2}\begin{Bmatrix} \lim_{n\to\infty} M(Ꞙ(g\varkappa_n, gy_n), Ꞙ(\varkappa, y), t) \\ + \lim_{n\to\infty} M(Ꞙ(\varkappa, y), Ꞙ\big(Ꞙ(\varkappa_n, y_n), Ꞙ(y_n, \varkappa_n)\big), t) \end{Bmatrix},$$

then, on using the continuity of Ꞙ on the right side of the above inequality, we get $\lim_{n\to\infty} M(Ꞙ(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq 1$, that is, $\lim_{n\to\infty} M(Ꞙ(g\varkappa_n, gy_n), g^2\varkappa_n, t) = 1$.

Similarly, we can obtain $\lim_{n\to\infty} M(Ꞙ(gy_n, g\varkappa_n), g^2y_n, t) = 1$.

We now show that $\lim_{n\to\infty} M\big(gꞘ(\varkappa_n, y_n), Ꞙ\big(Ꞙ(\varkappa_n, y_n), Ꞙ(y_n, \varkappa_n)\big), t\big) = 1$.

Since the pair (Ꞙ, g) is COM(B), we have

$$\lim_{n\to\infty} M(gꞘ(\varkappa_n, y_n), Ꞙ\big(Ꞙ(\varkappa_n, y_n), Ꞙ(y_n, \varkappa_n)\big), t) \geq \frac{1}{2} \begin{Bmatrix} \lim_{n\to\infty} M(gꞘ(\varkappa_n, y_n), g\varkappa, t) \\ + \lim_{n\to\infty} M(g\varkappa, g^2\varkappa_n, t) \end{Bmatrix}$$

then, on using the continuity of g on the right side of the above inequality, we obtain that $\lim_{n\to\infty} M(gꞘ(\varkappa_n, y_n), Ꞙ\big(Ꞙ(\varkappa_n, y_n), Ꞙ(y_n, \varkappa_n)\big), t) \geq 1$,

so that, $\lim_{n\to\infty} M(gꞘ(\varkappa_n, y_n), Ꞙ\big(Ꞙ(\varkappa_n, y_n), Ꞙ(y_n, \varkappa_n)\big), t) = 1$.

Similarly, we can obtain

$$\lim_{n\to\infty} M(gꞘ(y_n, \varkappa_n), Ꞙ(Ꞙ(y_n, \varkappa_n), Ꞙ(\varkappa_n, y_n)), t) = 1.$$

Hence, the pair (Ꞙ, g) is COM(A).

Likewise, it can be obtained that if both Ꞙ, g are continuous and the pair (Ꞙ, g) is COM(C), then it is COM(A).

Remark 8.2.5. In view of the above discussion, various relations between the variants of compatible mappings can be obtained easily under certain conditions. For example, we can easily observe that "If Ꞙ and g are both continuous, then the pair (Ꞙ, g) is COM(B) iff it is COM(C)".

Next, we discuss the relation between variants of compatible mappings and weakly compatible mappings.

Lemma 8.2.8. Let (Ҳ, M, ∗) be a FM-space and Ꞙ: Ҳ × Ҳ → Ҳ, g: Ҳ → Ҳ be two mappings. If the pair (Ꞙ, g) is compatible/ COM(A)/ COM(P)/ COM(B)/ COM(C)/ COM($A_Ꞙ$)/ COM(A_g), then, the pair (Ꞙ, g) is weakly compatible (w-compatible).

Proof. First, we shall show that if the pair (Ӻ, g) is compatible, then, it is also weakly compatible. For, let the pair (Ӻ, g) be compatible, then, we have

$$\lim_{n\to\infty} M(gӺ(\varkappa_n, y_n), Ӻ(g\varkappa_n, gy_n), ƫ) = 1$$

and $\lim_{n\to\infty} M(gӺ(y_n, \varkappa_n), Ӻ(gy_n, g\varkappa_n), ƫ) = 1$ for all ƫ > 0, whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ӿ such that $\lim_{n\to\infty} Ӻ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Ӻ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ӿ. Taking $\varkappa_n = ɑ$ and $y_n = ƀ$, we obtain that gɑ = Ӻ(ɑ, ƀ) and gƀ = Ӻ(ƀ, ɑ) implies that gӺ(ɑ, ƀ) = Ӻ(gɑ, gƀ) and gӺ(ƀ, ɑ) = Ӻ(gƀ, gɑ). Hence, every pair of compatible mappings is always weakly compatible (w-compatible).

Next, we shall show that if the pair (Ӻ, g) is COM(A), then, it is also weakly compatible. For, let the pair (Ӻ, g) be COM(A), then, we have

$$\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, ƫ) = 1, \quad \lim_{n\to\infty} M(Ӻ(gy_n, g\varkappa_n), g^2y_n, ƫ) = 1$$

and

$$\lim_{n\to\infty} M\big(gӺ(\varkappa_n, y_n), Ӻ\big(Ӻ(\varkappa_n, y_n), Ӻ(y_n, \varkappa_n)\big), ƫ\big) = 1,$$

$$\lim_{n\to\infty} M\big(gӺ(y_n, \varkappa_n), Ӻ\big(Ӻ(y_n, \varkappa_n), Ӻ(\varkappa_n, y_n)\big), ƫ\big) = 1,$$

whenever $\{\varkappa_n\}$ and $\{y_n\}$ are sequences in Ӿ such that $\lim_{n\to\infty} Ӻ(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} Ӻ(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ӿ. Taking $\varkappa_n = ɑ$ and $y_n = ƀ$, we obtain that gɑ = Ӻ(ɑ, ƀ) = ϰ and gƀ = Ӻ(ƀ, ɑ) = y. Also, then the condition

$$\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, ƫ) = 1 \text{ becomes } M(Ӻ(gɑ, gƀ), g^2ɑ, ƫ) = 1,$$

that is, $M(Ӻ(gɑ, gƀ), ggɑ, ƫ) = 1$, that is, $M(Ӻ(gɑ, gƀ), gӺ(ɑ, ƀ), ƫ) = 1$ which implies that Ӻ(gɑ, gƀ) = gӺ(ɑ, ƀ). Similarly, we can obtain that Ӻ(gƀ, gɑ) = gӺ(ƀ, ɑ). Therefore, gɑ = Ӻ(ɑ, ƀ) and gƀ = Ӻ(ƀ, ɑ) implies that Ӻ(gɑ, gƀ) = gӺ(ɑ, ƀ) and Ӻ(gƀ, gɑ) = gӺ(ƀ, ɑ). Hence, we can conclude that every pair of COM(A) is always weakly compatible (w-compatible).

Now, we show that if the pair (Ӻ, g) is COM(B), then, it is also a weakly compatible. For, let (Ӻ, g) of the mappings be COM(B), on taking $\varkappa_n = ɑ$ and $y_n = ƀ$, we obtain that gɑ = Ӻ(ɑ, ƀ) = ϰ and gƀ = Ӻ(ƀ, ɑ) = y. Then, the condition

$$\lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), g^2\varkappa_n, ƫ) \geq \frac{1}{2}\begin{cases} \lim_{n\to\infty} M(Ӻ(g\varkappa_n, gy_n), Ӻ(\varkappa, y), ƫ) \\ + \lim_{n\to\infty} M(Ӻ(\varkappa, y), Ӻ(Ӻ(\varkappa_n, y_n), Ӻ(y_n, \varkappa_n), ƫ) \end{cases}$$

in the definition of mappings of COM(B) becomes

$$M(Ӻ(gɑ, gƀ), g^2ɑ, ƫ) \geq \frac{1}{2}\{M(Ӻ(gɑ, gƀ), Ӻ(\varkappa, y), ƫ) + M(Ӻ(\varkappa, y), Ӻ(Ӻ(ɑ, ƀ), Ӻ(ƀ, ɑ)), ƫ)\},$$

that is,

$$\mathrm{M}(\mathrm{F}(g\alpha, gb), g\mathrm{F}(\alpha, b), t) \geq \frac{1}{2}\{\mathrm{M}(\mathrm{F}(g\alpha, gb), \mathrm{F}(\varkappa, y), t) + \mathrm{M}(\mathrm{F}(\varkappa, y), \mathrm{F}(\varkappa, y), t)\},$$

or $\mathrm{M}(\mathrm{F}(g\alpha, gb), g\mathrm{F}(\alpha, b), t) \geq \frac{1}{2}\{\mathrm{M}(\mathrm{F}(g\alpha, gb), \mathrm{F}(g\alpha, gb), t) + \mathrm{M}(\mathrm{F}(\varkappa, y), \mathrm{F}(\varkappa, y), t)\}$,

that is, $\mathrm{M}(\mathrm{F}(g\alpha, gb), g\mathrm{F}(\alpha, b), t) \geq 1$, so that $\mathrm{M}(\mathrm{F}(g\alpha, gb), g\mathrm{F}(\alpha, b), t) = 1$, for $t > 0$, hence, we get F(gα, gb) = gF(α, b). Similarly, we can obtain that F(gb, gα) = gF(b, α). Therefore, gα = F(α, b) and gb = F(b, α) implies that F(gα, gb) = gF(α, b) and F(gb, gα) = gF(b, α). Hence, we can conclude that every pair of COM(B) is always weakly compatible (w-compatible).

Likewise, we can prove that if the pair (F, g) is COM(P) or COM(C) or COM(A_F) or COM(A_g), then, it is weakly compatible (w-compatible).

The following example illustrates that the pair of weakly compatible mappings need not be compatible nor COM(A), COM(B), COM(P), COM(C), COM(A_F).

Example 8.2.8. Let X = [1, 20] and $*$ being any continuous t-norm. Define $\mathrm{M}(\varkappa, y, t) = e^{-|\varkappa - y|/t}$, for all $\varkappa$, y in X and $t > 0$. Then (X, M, $*$) is a FM-space. Let F: X × X → X and g: X → X be defined by

$$\mathrm{F}(\varkappa, y) = \begin{cases} 1, & \text{if } \varkappa = 1, \text{or } \varkappa > 4, y \in X \\ 5, & \text{if } 1 < \varkappa \leq 4, y \in X \end{cases} \quad \text{and} \quad g\varkappa = \begin{cases} 1, & \text{if } \varkappa = 1 \\ 12, & \text{if } 1 < \varkappa \leq 4 \\ \varkappa - 3, & \text{if } \varkappa > 4. \end{cases}$$

Then, the pair (F, g) is not compatible, since for the sequences $\{\varkappa_n\}$ and $\{y_n\}$ with $\varkappa_n = 4 + \frac{1}{2n}$ and $y_n = 4 + \frac{1}{2n+1}$ for $n \geq 1$, we have $\mathrm{F}(\varkappa_n, y_n) = 1$, $g\varkappa_n \to 1$, $\mathrm{F}(y_n, \varkappa_n) = 1$, $gy_n \to 1$, $\mathrm{M}(g\mathrm{F}(\varkappa_n, y_n), \mathrm{F}(g\varkappa_n, gy_n), t) = e^{-4/t} \nrightarrow 1$ as $n \to \infty$.

Also, for the above defined sequences $\{\varkappa_n\}$ and $\{y_n\}$, we have $\mathrm{M}(\mathrm{F}(g\varkappa_n, gy_n), g^2\varkappa_n, t) = e^{-7/t} \nrightarrow 1$ as $n \to \infty$, so that (F, g) is neither COM(A) nor COM(A_F). Next, we show that the pair (F, g) is not COM(B). Contrarily, let the pair (F, g) be COM(B), then, we must have

$$\lim_{n\to\infty} \mathrm{M}(\mathrm{F}(g\varkappa_n, gy_n), g^2\varkappa_n, t) \geq \frac{1}{2} \begin{cases} \lim_{n\to\infty} \mathrm{M}(\mathrm{F}(g\varkappa_n, gy_n), \mathrm{F}(\varkappa, y), t) \\ + \lim_{n\to\infty} \mathrm{M}(\mathrm{F}(\varkappa, y), \mathrm{F}(\mathrm{F}(\varkappa_n, y_n), \mathrm{F}(y_n, \varkappa_n)), t), \end{cases}$$

iff $e^{-7/t} \geq \frac{1}{2}\left(1 + e^{-4/t}\right)$ iff $2 \geq \left(e^{7/t} + e^{3/t}\right)$, which is not possible for $t > 0$. Hence, the pair (F, g) is not COM(B). Similarly, it is easy to obtain that the pair (F, g) is neither COM(C) nor COM(P). But the pair (F, g) is weakly compatible, since F and g commute at their only coupled coincidence point (1, 1).

Remark 8.2.6. (i) Since every pair of compatible mappings is weakly compatible, so that the pair (F, g) in Example 8.2.5 being compatible is also weakly compatible.

Also, the pair (Ғ, g) in Example 8.2.5 is not COM(A_g). Hence, Example 8.2.5 illustrates the fact that weakly compatible mappings need not be COM(A_g).

(ii) If the pair (Ғ, g) of mappings is commuting/ WC/ R-WC / R-WC($A_Ғ$)/ R-WC(A_g)/ R-WC(P), then, it is also weakly compatible (w-compatible). However, in general, the converse need not be true.

Next, we give the metrical version of the above definitions of variants of weakly commuting and compatible mappings.

Let (Ӽ, ḑ) be a metric space, then we define the following notions:

Definition 8.2.3. Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be two mappings. Then, the pair (Ғ, g) of mappings is said to be

(i) **Weakly commuting** (we write, **WC**), if for all ϰ, y in Ӽ, we have

$$ḑ(Ғ(gϰ, gy), gҒ(ϰ, y)) \leq ḑ(Ғ(ϰ, y), gϰ)$$

and
$$ḑ(Ғ(gy, gϰ), gҒ(y, ϰ)) \leq ḑ(Ғ(y, ϰ), gy);$$

(ii) **R-weakly commuting** (we write, **R-WC**), if there exists some R > 0 such that for all ϰ, y in Ӽ, we have

$$ḑ(Ғ(gϰ, gy), gҒ(ϰ, y)) \leq Rḑ(Ғ(ϰ, y), gϰ)$$

and
$$ḑ(Ғ(gy, gϰ), gҒ(y, ϰ)) \leq Rḑ(Ғ(y, ϰ), gy);$$

(iii) **R-weakly commuting of type ($A_Ғ$)** (we write, **R-WC($A_Ғ$)**), if there exists some R > 0 such that for all ϰ, y in Ӽ, we have

$$ḑ(Ғ(gϰ, gy), ggϰ) \leq Rḑ(Ғ(ϰ, y), gϰ)$$

and
$$ḑ(Ғ(gy, gϰ), ggy) \leq Rḑ(Ғ(y, ϰ), gy);$$

(iv) **R-weakly commuting of type (A_g)** (we write, **R-WC(A_g)**), if there exists some R > 0 such that for all ϰ, y in Ӽ, we have

$$ḑ(gҒ(ϰ, y), Ғ(Ғ(ϰ, y), Ғ(y, ϰ))) \leq Rḑ(Ғ(ϰ, y), gϰ)$$

and
$$ḑ(gҒ(y, ϰ), Ғ(Ғ(y, ϰ), Ғ(ϰ, y))) \leq Rḑ(Ғ(y, ϰ), gy);$$

(v) **R-weakly commuting of type (P)** (we write, **R-WC(P)**), if there exists some R > 0 such that for all ϰ, y in Ӽ, we have

$$ḑ(Ғ(Ғ(ϰ, y), Ғ(y, ϰ)), ggϰ) \leq Rḑ(Ғ(ϰ, y), gϰ)$$

and
$$ḑ(Ғ(Ғ(y, ϰ), Ғ(ϰ, y)), ggy) \leq Rḑ(Ғ(y, ϰ), gy).$$

Definition 8.2.4. Let Ғ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be two mappings. If whenever $\{ϰ_n\}$ and $\{y_n\}$ are sequences in Ӽ such that $\lim_{n\to\infty} Ғ(ϰ_n, y_n) = \lim_{n\to\infty} gϰ_n = ϰ$, $\lim_{n\to\infty} Ғ(y_n, ϰ_n) = \lim_{n\to\infty} gy_n = y$ for some ϰ, y in Ӽ, then, the pair (Ғ, g) is said to be

(i) **Compatible of type (A)** (we write, **COM(A)**), if

$$\lim_{n\to\infty} d(F(g\varkappa_n, gy_n), g^2\varkappa_n) = 0, \ \lim_{n\to\infty} d(F(gy_n, g\varkappa_n), g^2y_n) = 0$$

and

$$\lim_{n\to\infty} d\Big(gF(\varkappa_n, y_n), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\Big) = 0,$$

$$\lim_{n\to\infty} d\Big(gF(y_n, \varkappa_n), F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big)\Big) = 0;$$

(ii) **Compatible of type (B)** (we write, **COM(B)**), if

$$\lim_{n\to\infty} d(F(g\varkappa_n, gy_n), g^2\varkappa_n) \leq \frac{1}{2}\begin{cases} \lim_{n\to\infty} d\big(F(g\varkappa_n, gy_n), F(\varkappa, y)\big) \\ + \lim_{n\to\infty} d\Big(F(\varkappa, y), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\Big), \end{cases}$$

$$\lim_{n\to\infty} d(F(gy_n, g\varkappa_n), g^2y_n) \leq \frac{1}{2}\begin{cases} \lim_{n\to\infty} d(F(gy_n, gx_n), F(y, x)) \\ + \lim_{n\to\infty} d\Big(F(y, \varkappa), F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big)\Big), \end{cases}$$

and

$$\lim_{n\to\infty} d\Big(gF(\varkappa_n, y_n), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\Big) \leq \frac{1}{2}\begin{cases} \lim_{n\to\infty} d(gF(\varkappa_n, y_n), g\varkappa) \\ + \lim_{n\to\infty} d(g\varkappa, g^2\varkappa_n), \end{cases}$$

$$\lim_{n\to\infty} d\Big(gF(y_n, \varkappa_n), F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big)\Big) \leq \frac{1}{2}\begin{cases} \lim_{n\to\infty} d(gF(y_n, \varkappa_n), gy) \\ + \lim_{n\to\infty} d(gy, g^2y_n) \end{cases};$$

(iii) **Compatible of type (P)** (we write, **COM(P)**), if

$$\lim_{n\to\infty} d\Big(F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big), g^2\varkappa_n\Big) = 0,$$

$$\lim_{n\to\infty} d\Big(F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big), g^2y_n\Big) = 0;$$

(iv) **Compatible of type (C)** (we write, **COM(C)**), if

$$\lim_{n\to\infty} d(F(g\varkappa_n, gy_n), g^2\varkappa_n) \leq \frac{1}{3}\begin{cases} \lim_{n\to\infty} d\big(F(g\varkappa_n, gy_n), F(\varkappa, y)\big) \\ + \lim_{n\to\infty} d(F(\varkappa, y), g^2\varkappa_n) \\ + \lim_{n\to\infty} d\Big(F(\varkappa, y), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\Big), \end{cases}$$

$$\lim_{n\to\infty} d(F(gy_n, g\varkappa_n), g^2y_n) \leq \frac{1}{3}\begin{cases} \lim_{n\to\infty} d\big(F(gy_n, g\varkappa_n), F(y, \varkappa)\big) \\ + \lim_{n\to\infty} d(F(y, \varkappa), g^2y_n) \\ + \lim_{n\to\infty} d\Big(F(y, \varkappa), F\big(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\big)\Big), \end{cases}$$

and

$$\lim_{n\to\infty} d\Big(gF(\varkappa_n, y_n), F\big(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\big)\Big)$$

$$\leq \frac{1}{3}\begin{cases} \lim_{n\to\infty} d(gF(\varkappa_n, y_n), g\varkappa) \\ +\lim_{n\to\infty} d\left(g\varkappa, F\left(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\right)\right) \\ + \lim_{n\to\infty} d(g\varkappa, g^2\varkappa_n), \end{cases}$$

$$\lim_{n\to\infty} d\left(gF(y_n, \varkappa_n), F\left(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\right)\right)$$

$$\leq \frac{1}{3}\begin{cases} \lim_{n\to\infty} d(gF(y_n, \varkappa_n), gy) \\ +\lim_{n\to\infty} d\left(gy, F\left(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\right)\right); \\ + \lim_{n\to\infty} d(gy, g^2 y_n) \end{cases}$$

(v) **Compatible of type ($\mathbf{A_F}$)** (we write, **COM($\mathbf{A_F}$)**), if

$$\lim_{n\to\infty} d(F(g\varkappa_n, gy_n), gg\varkappa_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(F(gy_n, g\varkappa_n), ggy_n) = 0;$$

(vi) **Compatible of type ($\mathbf{A_g}$)** (we write, **COM($\mathbf{A_g}$)**), if

$$\lim_{n\to\infty} d\left(gF(\varkappa_n, y_n), F\left(F(\varkappa_n, y_n), F(y_n, \varkappa_n)\right)\right) = 0,$$

$$\lim_{n\to\infty} d\left(gF(y_n, \varkappa_n), F\left(F(y_n, \varkappa_n), F(\varkappa_n, y_n)\right)\right) = 0.$$

Remark 8.2.7. The relation between various mappings in the setup of fuzzy metric spaces established earlier also holds among the metrical versions of those mappings.

8.3. RESULTS FOR WEAKLY COMPATIBLE MAPPINGS

In this section, we give results for weakly compatible mappings and various mappings discussed in section 8.2, in the context of coupled fixed point theory in FM-spaces.

Let us denote by $\mathcal{W}$ the class of all continuous, non-decreasing functions ω: [0, 1] → [0, 1] with the property that $\omega(a) = 1$ iff $a = 1$. Also, denote by $\mathcal{V}$ the class of all continuous functions γ: [0, 1] → [0, 1].

Lemma 8.3.1. Let $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$. Assume that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$. Then, $\gamma(1) = 1$.

Proof. Let $\{a_n\} \subseteq (0, 1)$ be a non-decreasing sequence with $\lim_{n\to\infty} a_n = 1$. By hypothesis we have $\gamma(a_n) \geq \omega(a_n)$, $n \in \mathbb{N}$. Using the properties of γ and ω, we can obtain that $\gamma(1) \geq \omega(1) = 1$, which implies that $\gamma(1) = 1$. This completes the proof.

In order to give our main result, we shall first consider the following:

Let $(X, M, *)$ be a FM-space with (FM-6), $*$ being continuous t-norm of H-type. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four mappings such that $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$ and there exists $\phi \in \Phi_\phi$ such that

$$\omega\left(M\big(A(\varkappa, y), B(u, v), \phi(t)\big) * M\big(A(y, \varkappa), B(v, u), \phi(t)\big)\right)$$
$$\geq \gamma(M(S\varkappa, Tu, t) * M(Sy, Tv, t)), \quad (8.3.1)$$

for all $\varkappa$, y, u, v in X and $t > 0$, where $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$ such that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$. Since $A(X \times X) \subseteq T(X)$, so for arbitrary points $\varkappa_0$, y_0 in X, we can choose $\varkappa_1$, y_1 in X such that $T\varkappa_1 = A(\varkappa_0, y_0)$, $Ty_1 = A(y_0, \varkappa_0)$.

Again, since $B(X \times X) \subseteq S(X)$, we can choose $\varkappa_2$, y_2 in X such that $S\varkappa_2 = B(\varkappa_1, y_1)$ and $Sy_2 = B(y_1, \varkappa_1)$.

Continuing likewise, the sequences $\{z_n\}$ and $\{z_n'\}$ can be constructed in X such that

$$z_{2n+1} = A(\varkappa_{2n}, y_{2n}) = T(\varkappa_{2n+1}), z_{2n+2} = B(\varkappa_{2n+1}, y_{2n+1}) = S\varkappa_{2n+2} \quad (8.3.2)$$

and

$$z_{2n+1}' = A(y_{2n}, \varkappa_{2n}) = T(y_{2n+1}), z_{2n+2}' = B(y_{2n+1}, \varkappa_{2n+1}) = Sy_{2n+2}, \quad (8.3.3)$$

for all $n \geq 0$.

To prove the main result, we shall consider the following lemma:

Lemma 8.3.2. The sequences $\{z_n\}$ and $\{z_n'\}$ defined by (8.3.2) and (8.3.3), respectively are Cauchy sequences in X.

Proof. Since $*$ is a t-norm of H-type, for $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}. \quad (8.3.4)$$

By (FM-6), there exists $t_0 > 0$ such that

$$M(S\varkappa_0, T\varkappa_1, t_0) \geq (1-\varrho) \quad \text{and} \quad M(Sy_0, Ty_1, t_0) \geq (1-\varrho). \quad (8.3.5)$$

Also, since $\phi \in \Phi_\phi$, by (ϕ-3), for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{j=n_0}^{\infty} \phi^j(t_0). \quad (8.3.6)$$

Using (8.3.1), we can get

$$\omega\left(M\big(z_1, z_2, \phi(t_0)\big) * M\left(z_1', z_2', \phi(t_0)\right)\right)$$
$$= \omega\left(M\big(A(\varkappa_0, y_0), B(\varkappa_1, y_1), \phi(t_0)\big) * M\big(A(y_0, \varkappa_0), B(y_1, \varkappa_1), \phi(t_0)\big)\right)$$
$$\geq \gamma\left(M\big(S\varkappa_0, T\varkappa_1, t_0\big) * M(Sy_0, Ty_1, t_0)\right)$$
$$\geq \omega\left(M\big(S\varkappa_0, T\varkappa_1, t_0\big) * M(Sy_0, Ty_1, t_0)\right),$$

then, using the monotone property of ω, we get

$$M\big(z_1, z_2, \phi(t_0)\big) * M\left(z_1', z_2', \phi(t_0)\right) \geq M\big(S\varkappa_0, T\varkappa_1, t_0\big) * M(Sy_0, Ty_1, t_0).$$

Again using (8.3.1), we get

$$\omega\left(M(z_2, z_3, \phi^2(t_0)) * M\left(z_2', z_3', \phi^2(t_0)\right)\right)$$
$$= \omega\left(M(B(\varkappa_1, y_1), A(\varkappa_2, y_2), \phi^2(t_0)) * M(B(y_1, \varkappa_1), A(y_2, \varkappa_2), \phi^2(t_0))\right)$$
$$\geq \gamma(M(S\varkappa_2, T\varkappa_1, \phi(t_0)) * M(Sy_2, Ty_1, \phi(t_0)))$$
$$\geq \omega(M(S\varkappa_2, T\varkappa_1, \phi(t_0)) * M(Sy_2, Ty_1, \phi(t_0)))$$
$$= \omega(M(z_2, z_1, \phi(t_0)) * M(z_2', z_1', \phi(t_0)))$$

then, by monotone property of ω, we get

$$M(z_2, z_3, \phi^2(t_0)) * M\left(z_2', z_3', \phi^2(t_0)\right) \geq M(z_1, z_2, \phi(t_0)) * M\left(z_1', z_2', \phi(t_0)\right).$$

Similarly,

$$M(z_3, z_4, \phi^3(t_0)) * M\left(z_3', z_4', \phi^3(t_0)\right) \geq M(z_2, z_3, \phi^2(t_0)) * M\left(z_2', z_3', \phi^2(t_0)\right).$$

Continuing likewise, for all n > 0, we can obtain

$$M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * M\left(z_{n+1}', z_{n+2}', \phi^{n+1}(t_0)\right) \geq M(z_n, z_{n+1}, \phi^n(t_0)) * M\left(z_n', z_{n+1}', \phi^n(t_0)\right),$$

which implies that

$$M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * M\left(z_{n+1}', z_{n+2}', \phi^{n+1}(t_0)\right) \geq M(S\varkappa_0, T\varkappa_1, t_0) * M(Sy_0, Ty_1, t_0).$$

Now, using (8.3.4) − (8.3.6), for m > n ≥ n_0, we get

$$M(z_n, z_m, t) * M(z_n', z_m', t)$$
$$\geq M\left(z_n, z_m, \textstyle\sum_{j=n_0}^{\infty} \phi^j(t_0)\right) * M\left(z_n', z_m', \textstyle\sum_{j=n_0}^{\infty} \phi^j(t_0)\right)$$
$$\geq M\left(z_n, z_m, \textstyle\sum_{j=n}^{m-1} \phi^j(t_0)\right) * M\left(z_n', z_m', \textstyle\sum_{j=n}^{m-1} \phi^j(t_0)\right)$$
$$\geq \left[M(z_n, z_{n+1}, \phi^n(t_0)) * M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * \ldots * M(z_{m-1}, z_m, \phi^{m-1}(t_0))\right]$$
$$* \left[M\left(z_n', z_{n+1}', \phi^n(t_0)\right) * M\left(z_{n+1}', z_{n+2}', \phi^{n+1}(t_0)\right) * \ldots * M\left(z_{m-1}', z_m', \phi^{m-1}(t_0)\right)\right]$$
$$= \left[M(z_n, z_{n+1}, \phi^n(t_0)) * M\left(z_n', z_{n+1}', \phi^n(t_0)\right)\right]$$
$$* \left[M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * M\left(z_{n+1}', z_{n+2}', \phi^{n+1}(t_0)\right)\right]$$
$$\ldots$$
$$* \left[M(z_{m-1}, z_m, \phi^{m-1}(t_0)) * M\left(z_{m-1}', z_m', \phi^{m-1}(t_0)\right)\right]$$
$$\geq [M(S\varkappa_0, T\varkappa_1, t_0) * M(Sy_0, Ty_1, t_0)] * [M(S\varkappa_0, T\varkappa_1, t_0) * M(Sy_0, Ty_1, t_0)] * \ldots$$
$$* [M(S\varkappa_0, T\varkappa_1, t_0) * M(Sy_0, Ty_1, t_0)]$$
$$\geq \underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{2(m-n)} \geq (1-\sigma),$$

which implies that $\mathrm{M}(z_n, z_m, t) * \mathrm{M}(z_n', z_m', t) \geq (1-\sigma)$, for all m, n ∈ ℕ with m > n > n_0 and t > 0. So that $\{z_n\}$ and $\{z_n'\}$ both are Cauchy sequences in X.

Now, we give our main result as follows:

Theorem 8.3.1. Let (X, M, ∗) be a FM-space with (FM-6), ∗ being continuous t-norm of H-type. Let A, B: X × X → X and S, T: X → X be four mappings such that A(X × X) ⊆ T(X), B(X × X) ⊆ S(X) and there exists some $\phi \in \Phi_\phi$ such that (8.3.1) holds for all ϰ, y, u, v in X and t > 0 with $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$ such that $\gamma(a) \geq \omega(a)$ for a ∈ [0, 1]. Suppose that the pairs (A, S) and (B, T) are weakly compatible, one of the subspaces A(X × X) or T(X) and one of B(X × X) or S(X) are complete. Then, there exists a unique point α in X such that A(α, α) = Sα = α = Tα = B(α, α).

Proof. By Lemma 8.3.2, both the sequences $\{z_n\}$ and $\{z_n'\}$ defined respectively by (8.3.2) and (8.3.3) are Cauchy sequences. The proof is divided into four steps as follows:

Step 1. We assert the existence of some α, β in X such that

Tα = B(α, β), Tβ = B(β, α) and Sα = A(α, β), Sβ = A(β, α).

W.L.O.G., assume that the subspaces T(X) and S(X) are complete. Since $\{z_{2n+1}\}$, $\{z_{2n+2}\}$ and $\{z_{2n+1}'\}$, $\{z_{2n+2}'\}$ are the sub-sequences of the Cauchy sequences $\{z_n\}$ and $\{z_n'\}$ respectively, so they are also Cauchy sequences. By completeness of T(X), there exists α, β in T(X) ⊆ X such that $\{z_{2n+1}\} \to \alpha$ and $\{z_{2n+1}'\} \to \beta$ as n → ∞. By convergence of sub-sequences $\{z_{2n+1}\}$ and $\{z_{2n+1}'\}$, it is easy to establish the convergence of the original Cauchy sequences $\{z_n\}$ and $\{z_n'\}$ respectively, such that $\{z_n\} \to \alpha$ and $\{z_n'\} \to \beta$ as n → ∞. Consequently, it follows that the sequences $\{z_{2n+1}\}$, $\{z_{2n+2}\}$, $\{z_n\}$ converges to α and $\{z_{2n+1}'\}$, $\{z_{2n+2}'\}$, $\{z_n'\}$ converges to β. Since α, β ∈ T(X), there exist some p, q in X such that Tp = α, Tq = β, so that, we have

$$\lim_{n\to\infty} z_{2n+1} = \lim_{n\to\infty} \mathrm{A}(\varkappa_{2n}, y_{2n}) = \lim_{n\to\infty} \mathrm{T}\varkappa_{2n+1} = \alpha = \mathrm{T}p,$$

$$\lim_{n\to\infty} z_{2n+2} = \lim_{n\to\infty} \mathrm{B}(\varkappa_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} \mathrm{S}\varkappa_{2n+2} = \alpha = \mathrm{T}p,$$

$$\lim_{n\to\infty} z_{2n+1}' = \lim_{n\to\infty} \mathrm{A}(y_{2n}, \varkappa_{2n}) = \lim_{n\to\infty} \mathrm{T}y_{2n+1} = \beta = \mathrm{T}q$$

and $$\lim_{n\to\infty} z_{2n+2}' = \lim_{n\to\infty} \mathrm{B}(y_{2n+1}, \varkappa_{2n+1}) = \lim_{n\to\infty} \mathrm{S}y_{2n+2} = \beta = \mathrm{T}q.$$

By (8.3.1), we can obtain

$$\omega\Big(\mathrm{M}\big(\mathrm{A}(\varkappa_{2n}, y_{2n}), \mathrm{B}(p, q), \phi(t)\big) * \mathrm{M}\big(\mathrm{A}(y_{2n}, \varkappa_{2n}), \mathrm{B}(q, p), \phi(t)\big)\Big)$$

$\geq \gamma\big(M(Ş\varkappa_{2n}, Ţp, t) * M(Şy_{2n}, Ţq, t)\big)$

$\geq \omega\big(M(Ş\varkappa_{2n}, Ţp, t) * M(Şy_{2n}, Ţq, t)\big)$,

then, using the monotone property of ω, we get

$$M\big(A(\varkappa_{2n}, y_{2n}), B(p, q), \phi(t)\big) * M\big(A(y_{2n}, \varkappa_{2n}), B(q, p), \phi(t)\big)$$
$$\geq M(Ş\varkappa_{2n}, Ţp, t) * M(Şy_{2n}, Ţq, t),$$

then, on letting $n \to \infty$, we obtain that

$$M\big(Ţp, B(p, q), \phi(t)\big) * M\big(Ţq, B(q, p), \phi(t)\big) \geq 1,$$

which implies that $Ţp = B(p, q) = \alpha$ and $Ţq = B(q, p) = \beta$. As the pair $(B, Ţ)$ is weakly compatible, so that $Ţp = B(p, q) = \alpha$ implies that $Ţ\alpha = B(\alpha, \beta)$. Similarly, we can get $Ţ\beta = B(\beta, \alpha)$. Also, since $Ş(X)$ is complete, so $\alpha, \beta \in Ş(X)$, which implies the existence of some r, s in X such that $Şr = \alpha$, $Şs = \beta$.

Again using (8.3.1), we obtain that

$\omega\Big(M\big(A(r, s), B(\varkappa_{2n+1}, y_{2n+1}), \phi(t)\big) * M\big(A(s, r), B(y_{2n+1}, \varkappa_{2n+1}), \phi(t)\big)\Big)$

$\geq \gamma\big(M(Şr, Ţ\varkappa_{2n+1}, t) * M(Şs, Ţy_{2n+1}, t)\big)$,

then, on letting $n \to \infty$ and using the continuity of ω, γ we can obtain

$$\omega\Big(M\big(A(r, s), \alpha, \phi(t)\big) * M\big(A(s, r), \beta, \phi(t)\big)\Big) \geq \gamma(1) = 1,$$

which implies that $A(r, s) = \alpha = Şr$ and $A(s, r) = \beta = Şs$. Since, the pair $(A, Ş)$ is weakly compatible, it follows that $A(\alpha, \beta) = Ş\alpha$ and $A(\beta, \alpha) = Ş\beta$.

Step 2. Next, we show that $Ş\alpha = Ţ\alpha$ and $Ş\beta = Ţ\beta$.

Since $*$ is a t-norm of H-type, for $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \dots * (1-\varrho)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $t_0 > 0$ such that

$$M(Ş\alpha, Ţ\alpha, t_0) \geq (1-\varrho) \text{ and } M(Ş\beta, Ţ\beta, t_0) \geq (1-\varrho).$$

Also, by (ϕ-3), for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{j=n_0}^{\infty} \phi^j(t_0)$.

By (8.3.1), we obtain that

$\omega\Big(M\big(Ş\alpha, Ţ\alpha, \phi(t_0)\big) * M\big(Ş\beta, Ţ\beta, \phi(t_0)\big)\Big)$

$= \omega\Big(M\big(A(\alpha, \beta), B(\alpha, \beta), \phi(t_0)\big) * M\big(A(\beta, \alpha), B(\beta, \alpha), \phi(t_0)\big)\Big)$

$\geq \gamma\big(M(Ş\alpha, Ţ\alpha, t_0) * M(Ş\beta, Ţ\beta, t_0)\big)$.

Since $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$, by last inequality, we get

$\omega\Big(M\big(Ş\alpha, Ţ\alpha, \phi(t_0)\big) * M\big(Ş\beta, Ţ\beta, \phi(t_0)\big)\Big)$

$$\geq \omega\big(\mathrm{M}(\c{S}\alpha, \c{T}\alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \c{T}\beta, \mathfrak{t}_0)\big).$$

By the monotone property of ω, we obtain that

$$\mathrm{M}\big(\c{S}\alpha, \c{T}\alpha, \phi(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \c{T}\beta, \phi(\mathfrak{t}_0)\big)$$
$$\geq \mathrm{M}(\c{S}\alpha, \c{T}\alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \c{T}\beta, \mathfrak{t}_0).$$

Reasoning as above, in general for all n ≥ 1, we can obtain

$$\mathrm{M}\big(\c{S}\alpha, \c{T}\alpha, \phi^{\mathrm{n}}(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \c{T}\beta, \phi^{\mathrm{n}}(\mathfrak{t}_0)\big)$$
$$\geq \mathrm{M}(\c{S}\alpha, \c{T}\alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \c{T}\beta, \mathfrak{t}_0).$$

Thus, for $\sigma > 0$ and $\mathfrak{f} > 0$, we have

$$\mathrm{M}(\c{S}\alpha, \c{T}\alpha, \mathfrak{f}) * \mathrm{M}(\c{S}\beta, \c{T}\beta, \mathfrak{f})$$
$$\geq \mathrm{M}\big(\c{S}\alpha, \c{T}\alpha, \textstyle\sum_{\mathrm{j}=\mathrm{n}_0}^{\infty} \phi^{\mathrm{j}}(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \c{T}\beta, \textstyle\sum_{\mathrm{j}=\mathrm{n}_0}^{\infty} \phi^{\mathrm{j}}(\mathfrak{t}_0)\big)$$
$$\geq \mathrm{M}\big(\c{S}\alpha, \c{T}\alpha, \phi^{\mathrm{n}_0}(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \c{T}\beta, \phi^{\mathrm{n}_0}(\mathfrak{t}_0)\big)$$
$$\geq \mathrm{M}(\c{S}\alpha, \c{T}\alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \c{T}\beta, \mathfrak{t}_0) \geq (1-\varrho) * (1-\varrho) \geq (1-\sigma).$$

Hence, $\c{S}\alpha = \c{T}\alpha$ and $\c{S}\beta = \c{T}\beta$.

Therefore, $\c{S}\alpha = \text{Ⱥ}(\alpha, \beta) = \text{Ƀ}(\alpha, \beta) = \c{T}\alpha$ and $\c{S}\beta = \text{Ⱥ}(\beta, \alpha) = \text{Ƀ}(\beta, \alpha) = \c{T}\beta$.

Step 3. We next show that $\c{S}\alpha = \alpha$ and $\c{S}\beta = \beta$.

Since $*$ is a t-norm of H-type, for any $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{\mathrm{i}} \geq (1-\sigma) \text{ for all } \mathrm{i} \in \mathbb{N}.$$

By (FM-6), there exists $\mathfrak{t}_0 > 0$ such that

$$\mathrm{M}(\alpha, \c{S}\alpha, \mathfrak{t}_0) \geq (1-\varrho) \text{ and } \mathrm{M}(\beta, \c{S}\beta, \mathfrak{t}_0) \geq (1-\varrho).$$

Also, since $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{f} > 0$, there exists $\mathrm{n}_0 \in \mathbb{N}$ such that

$$\mathfrak{f} > \textstyle\sum_{\mathrm{j}=\mathrm{n}_0}^{\infty} \phi^{\mathrm{j}}(\mathfrak{t}_0).$$

By (8.3.1), we have

$$\omega\left(\mathrm{M}\big(\c{S}\alpha, \alpha, \phi(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \beta, \phi(\mathfrak{t}_0)\big)\right)$$
$$= \omega\left(\mathrm{M}\big(\text{Ⱥ}(\alpha, \beta), \text{Ƀ}(\mathrm{p}, \mathrm{q}), \phi(\mathfrak{t}_0)\big) * \mathrm{M}\big(\text{Ⱥ}(\beta, \alpha), \text{Ƀ}(\mathrm{q}, \mathrm{p}), \phi(\mathfrak{t}_0)\big)\right)$$
$$\geq \gamma\big(\mathrm{M}(\c{S}\alpha, \c{T}\mathrm{p}, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \c{T}\mathrm{q}, \mathfrak{t}_0)\big)$$
$$= \gamma\big(\mathrm{M}(\c{S}\alpha, \alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \beta, \mathfrak{t}_0)\big),$$

then, using the fact that $\gamma(\mathrm{a}) \geq \omega(\mathrm{a})$ for $\mathrm{a} \in [0, 1]$ and the monotone property of ω, we obtain that

$$\mathrm{M}\big(\c{S}\alpha, \alpha, \phi(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \beta, \phi(\mathfrak{t}_0)\big) \geq \mathrm{M}(\c{S}\alpha, \alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \beta, \mathfrak{t}_0).$$

In general, for all n ≥ 1, we obtain that

$$\mathrm{M}\big(\c{S}\alpha, \alpha, \phi^{\mathrm{n}}(\mathfrak{t}_0)\big) * \mathrm{M}\big(\c{S}\beta, \beta, \phi^{\mathrm{n}}(\mathfrak{t}_0)\big) \geq \mathrm{M}(\c{S}\alpha, \alpha, \mathfrak{t}_0) * \mathrm{M}(\c{S}\beta, \beta, \mathfrak{t}_0).$$

Now, for any $\sigma > 0$ and for all $\mathfrak{t} > 0$, we have

$\mathsf{M}(\c{S}\alpha, \alpha, \mathfrak{t}) * \mathsf{M}(\c{S}\beta, \beta, \mathfrak{t})$

$\geq \mathsf{M}\left(\c{S}\alpha, \alpha, \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)\right) * \mathsf{M}\left(\c{S}\beta, \beta, \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)\right)$

$\geq \mathsf{M}\left(\c{S}\alpha, \alpha, \phi^{n_0}(\mathfrak{t}_0)\right) * \mathsf{M}\left(\c{S}\beta, \beta, \phi^{n_0}(\mathfrak{t}_0)\right) \geq \mathsf{M}(\c{S}\alpha, \alpha, \mathfrak{t}_0) * \mathsf{M}(\c{S}\beta, \beta, \mathfrak{t}_0)$

$\geq (1 - \varrho) * (1 - \varrho) \geq (1 - \sigma)$.

Therefore, $\c{S}\alpha = \alpha$ and $\c{S}\beta = \beta$. Thus, we have $\mathsf{B}(\alpha, \beta) = \c{S}\alpha = \alpha = \c{T}\alpha = \mathsf{A}(\alpha, \beta)$ and $\mathsf{B}(\beta, \alpha) = \c{S}\beta = \beta = \c{T}\beta = \mathsf{A}(\beta, \alpha)$.

Step 4. We now show that $\alpha = \beta$.

Since $*$ is a t-norm of H-type, for any $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1 - \varrho) * (1 - \varrho) * \ldots * (1 - \varrho)}_{i} \geq (1 - \sigma) \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $\mathfrak{t}_0 > 0$ such that

$$\mathsf{M}(\alpha, \beta, \mathfrak{t}_0) \geq (1 - \varrho).$$

Also, since $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{t} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathfrak{t} > \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0).$$

By (8.3.1), we have

$\omega\left(\mathsf{M}\left(\alpha, \beta, \phi(\mathfrak{t}_0)\right) * \mathsf{M}\left(\beta, \alpha, \phi(\mathfrak{t}_0)\right)\right)$

$= \omega\left(\mathsf{M}\left(\mathsf{A}(p, q), \mathsf{B}(q, p), \phi(\mathfrak{t}_0)\right) * \mathsf{M}\left(\mathsf{A}(q, p), \mathsf{B}(p, q), \phi(\mathfrak{t}_0)\right)\right)$

$\geq \gamma\left(\mathsf{M}(\c{S}p, \c{T}q, \mathfrak{t}_0) * \mathsf{M}(\c{S}q, \c{T}p, \mathfrak{t}_0)\right)$

$= \gamma\left(\mathsf{M}(\alpha, \beta, \mathfrak{t}_0) * \mathsf{M}(\beta, \alpha, \mathfrak{t}_0)\right)$,

then, using the fact that $\gamma(a) \geq \omega(a)$ for $a \in [0, 1]$ and the monotone property of ω, we get

$$\mathsf{M}\left(\alpha, \beta, \phi(\mathfrak{t}_0)\right) * \mathsf{M}\left(\beta, \alpha, \phi(\mathfrak{t}_0)\right) \geq \mathsf{M}(\alpha, \beta, \mathfrak{t}_0) * \mathsf{M}(\beta, \alpha, \mathfrak{t}_0).$$

In general, for all $n \geq 1$, we obtain that

$$\mathsf{M}\left(\alpha, \beta, \phi^n(\mathfrak{t}_0)\right) * \mathsf{M}\left(\beta, \alpha, \phi^n(\mathfrak{t}_0)\right) \geq \mathsf{M}(\alpha, \beta, \mathfrak{t}_0) * \mathsf{M}(\beta, \alpha, \mathfrak{t}_0).$$

Then, for $\sigma > 0$ and for all $\mathfrak{t} > 0$, we have

$\mathsf{M}(\alpha, \beta, \mathfrak{t}) * \mathsf{M}(\beta, \alpha, \mathfrak{t})$

$\geq \mathsf{M}\left(\alpha, \beta, \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)\right) * \mathsf{M}\left(\beta, \alpha, \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)\right)$

$\geq \mathsf{M}\left(\alpha, \beta, \phi^{n_0}(\mathfrak{t}_0)\right) * \mathsf{M}\left(\beta, \alpha, \phi^{n_0}(\mathfrak{t}_0)\right)$

$\geq \mathsf{M}(\alpha, \beta, \mathfrak{t}_0) * \mathsf{M}(\beta, \alpha, \mathfrak{t}_0)$

$\geq (1 - \varrho) * (1 - \varrho) \geq (1 - \sigma)$,

which implies that $\alpha = \beta$.

Hence, there exists some point α in X such that $A(\alpha, \alpha) = T\alpha = \alpha = S\alpha = B(\alpha, \alpha)$. Uniqueness of the point α follows immediately by using (8.3.1).

Theorem 8.3.2. Theorem 8.3.1 remains true if the 'weakly compatible property' is replaced by any one of the properties (retaining the rest of the hypotheses):

(i) Compatibility;

(ii) COM(A);

(iii) COM(P);

(iv) COM(B);

(v) COM(C);

(vi) COM(A_F);

(vii) COM(A_g).

Proof. Using Lemma 8.2.8, the proof follows immediately.

Theorem 8.3.3. Theorem 8.3.1 remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypothesis) of the properties:

(i) Commuting;

(ii) WC;

(iii) R-WC;

(iv) R-WC(A_F);

(v) R-WC(A_g);

(vi) R-WC(P).

Proof. Using the (ii) part of Remark 8.2.6, the proof follows immediately.

8.4. PROPERTY: (E.A.), (CLRg), COMMON PROPERTY (E.A.) AND (CLR$_{ST}$)

This section deals with the notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR$_{ST}$) property in coupled fixed point theory. Further, utilizing these notions, the results of Hu [146], Hu et al. [147] and Jain et al. [63] (that is, Theorems 8.1.1, 8.1.2 and 8.1.3, respectively) are also generalized.

We now discuss the following notions:

Definition 8.4.1. Let (X, d) be a metric space and $F: X \times X \to X$, $g: X \to X$ be two mappings. Then, the pair (F, g) is said to satisfy **property (E.A.)**, if there exist sequences $\{\varkappa_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = \varkappa$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = y$ for some $\varkappa$, y in X.

The fuzzy metric analogue of Definition 8.4.1 is as follows:

"Let (X, M, ∗) be a FM-space and F: X × X → X, g: X → X be two mappings. Then, the pair (F, g) is said to satisfy **property (E.A.)**, if there exist sequences $\{\varkappa_n\}$ and $\{y_n\}$ in X such that $\{F(\varkappa_n, y_n)\}$, $\{g\varkappa_n\}$ converges to $\varkappa$ and $\{F(y_n, \varkappa_n)\}$, $\{gy_n\}$ converges to y for some $\varkappa$, y in X, w.r.t convergence in (X, M, ∗)".

Definition 8.4.2. Let (X, d) be a metric space and F: X × X → X, g: X → X be two mappings. Then, the pair (F, g) is said to satisfy **(CLRg) property**, if there exist sequences $\{\varkappa_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} F(\varkappa_n, y_n) = \lim_{n\to\infty} g\varkappa_n = gp$, $\lim_{n\to\infty} F(y_n, \varkappa_n) = \lim_{n\to\infty} gy_n = gq$ for some p, q in X.

The fuzzy metric analogue of Definition 8.4.2 is as follows:

"Let (X, M, ∗) be a FM-space and F: X × X → X, g: X → X be two mappings. Then, the pair (F, g) is said to satisfy **(CLRg) property**, if there exist sequences $\{\varkappa_n\}$ and $\{y_n\}$ in X such that $\{F(\varkappa_n, y_n)\}$, $\{g\varkappa_n\}$ converges to gp and $\{F(y_n, \varkappa_n)\}$, $\{gy_n\}$ converges to gq for some p, q in X, w.r.t convergence in (X, M, ∗)".

We now extend Definition 8.4.1 under the following notion:

Definition 8.4.3. Let (X, d) be a metric space and A, B: X × X → X and S, T: X → X be the mappings. Then, the pairs (A, S) and (B, T) are said to share **common property (E.A.)**, if there exist sequences $\{\varkappa_n\}$, $\{y_n\}$ and $\{p_n\}$, $\{q_n\}$ in X such that

$$\lim_{n\to\infty} A(\varkappa_n, y_n) = \lim_{n\to\infty} B(p_n, q_n) = \lim_{n\to\infty} Tp_n = \lim_{n\to\infty} S\varkappa_n = a,$$

$$\lim_{n\to\infty} A(y_n, \varkappa_n) = \lim_{n\to\infty} B(q_n, p_n) = \lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Sy_n = b \text{ for some } a, b \text{ in } X.$$

The fuzzy metric analogue of Definition 8.4.3 is given as:

"Let (X, M, ∗) be a FM-space and A, B: X × X → X and S, T: X → X be the mappings. Then, the pairs (A, S) and (B, T) are said to share **common property (E.A.)**, if there exists sequences $\{\varkappa_n\}$, $\{y_n\}$ and $\{p_n\}$, $\{q_n\}$ in X such that $\{A(\varkappa_n, y_n)\}$, $\{B(p_n, q_n)\}$, $\{Tp_n\}$, $\{S\varkappa_n\}$ converges to a and $\{A(y_n, \varkappa_n)\}$, $\{B(q_n, p_n)\}$, $\{Tq_n\}$, $\{Sy_n\}$ converges to b for some a, b ∈ X, w.r.t convergence in (X, M, ∗)".

Remark 8.4.1. On taking A = B = F and S = T = g in Definition 8.4.3, we obtain Definition 8.4.1.

Next, we define the notion of (CLR_{ST}) property in coupled fixed point theory.

Definition 8.4.4. Let (X, d) be a metric space and A, B: X × X → X and S, T: X → X be the mappings. Then, the pairs (A, S) and (B, T) are said to satisfy **(CLR_{ST}) property**, if there exist sequences $\{\varkappa_n\}$, $\{y_n\}$ and $\{p_n\}$, $\{q_n\}$ in X such that

$$\lim_{n\to\infty} Ꞥ(\varkappa_n, y_n) = \lim_{n\to\infty} Ƀ(ꞵ_n, ꝗ_n) = \lim_{n\to\infty} Ţꞵ_n = \lim_{n\to\infty} Ş\varkappa_n = ɑ,$$

$$\lim_{n\to\infty} Ꞥ(y_n, \varkappa_n) = \lim_{n\to\infty} Ƀ(ꝗ_n, ꞵ_n) = \lim_{n\to\infty} Ţꝗ_n = \lim_{n\to\infty} Şy_n = ƀ,$$

for some ɑ, ƀ ∈ Ş(Ҳ) ∩ Ţ(Ҳ).

Following is the fuzzy metric analogue of Definition 8.4.4:

"Let (Ҳ, Ϻ, ∗) be a FϺ-space and Ꞥ, Ƀ: Ҳ × Ҳ → Ҳ and Ş, Ţ: Ҳ → Ҳ be the mappings. The pairs (Ꞥ, Ş) and (Ƀ, Ţ) are said to satisfy **($CLR_{ŞŢ}$) property**, if there exists sequences $\{\varkappa_n\}$, $\{y_n\}$ and $\{ꞵ_n\}$, $\{ꝗ_n\}$ in Ҳ such that $\{Ꞥ(\varkappa_n, y_n)\}$, $\{Ƀ(ꞵ_n, ꝗ_n)\}$, $\{Ţꞵ_n\}$, $\{Ş\varkappa_n\}$ converges to ɑ and $\{Ꞥ(y_n, \varkappa_n)\}$, $\{Ƀ(ꝗ_n, ꞵ_n)\}$, $\{Ţꝗ_n\}$, $\{Şy_n\}$ converges to ƀ for some ɑ, ƀ ∈ Ş(Ҳ) ∩ Ţ(Ҳ), w.r.t convergence in (Ҳ, Ϻ, ∗)".

Remark 8.4.2. Taking Ꞥ = Ƀ = Ғ and Ş = Ţ = ꞡ in Definition 8.4.4, we obtain Definition 8.4.2.

We now give our main result that generalizes Theorems 8.1.1 and 8.1.2.

Theorem 8.4.1. Let (Ҳ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being continuous t-norm of H-type. Let Ғ: Ҳ × Ҳ → Ҳ and ꞡ: Ҳ → Ҳ be two mappings and there exists $\phi \in \Phi_\phi$ satisfying (8.1.1). Suppose that the pair (Ғ, ꞡ) is weakly compatible and satisfies (CLRꞡ) property. Then, there exists a unique ϰ in Ҳ such that Ғ(ϰ, ϰ) = ϰ = ꞡϰ.

Proof. As (Ғ, ꞡ) satisfies (CLRꞡ) property, there exist sequences $\{\varkappa_n\}$, $\{y_n\}$ in Ҳ such that $\lim_{n\to\infty} Ғ(\varkappa_n, y_n) = \lim_{n\to\infty} ꞡ\varkappa_n = ꞡp$, $\lim_{n\to\infty} Ғ(y_n, \varkappa_n) = \lim_{n\to\infty} ꞡy_n = ꞡq$ for some p, q in Ҳ. The proof consists of the following steps:

Step 1. We assert that Ғ and ꞡ have a coupled coincidence point.

By (8.1.1), for ƭ > 0, we have

$$Ϻ(Ғ(\varkappa_n, y_n), Ғ(p, q), ƭ) \geq Ϻ(Ғ(\varkappa_n, y_n), Ғ(p, q), \phi(ƭ))$$
$$\geq Ϻ(ꞡ\varkappa_n, ꞡp, ƭ) * Ϻ(ꞡy_n, ꞡq, ƭ),$$

then, on letting n → ∞, we get Ϻ(ꞡp, Ғ(p, q), ƭ) = 1, that is, Ғ(p, q) = ꞡp = ϰ (say). Likewise, we can get Ғ(q, p) = ꞡq = y (say). As (Ғ, ꞡ) is weakly compatible, so, we can obtain that ꞡҒ(p, q) = Ғ(ꞡp, ꞡq) and ꞡҒ(q, p) = Ғ(ꞡq, ꞡp), that is, ꞡϰ = Ғ(ϰ, y) and ꞡy = Ғ(y, ϰ).

Step 2. We show that ꞡϰ = ϰ, ꞡy = y.

Since ∗ is a t-norm of H-type, for any $\sigma > 0$, there exists ϱ > 0 such that

$$\underbrace{(1-ϱ) * (1-ϱ) * \ldots * (1-ϱ)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FϺ-6), there exists $ƭ_0 > 0$ such that

$\mathrm{M}(g\varkappa, \varkappa, t_0) \geq (1-\varrho)$ and $\mathrm{M}(gy, y, t_0) \geq (1-\varrho)$.

As $\phi \in \Phi_\phi$, by (ϕ-3), for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{j=n_0}^{\infty} \phi^j(t_0)$.

Using (8.1.1), we have

$$\begin{aligned}
&\mathrm{M}(g\varkappa, \varkappa, \phi(t_0)) \\
&\quad = \mathrm{M}(F(\varkappa, y), F(p, q), \phi(t_0)) \\
&\quad \geq \mathrm{M}(g\varkappa, gp, t_0) * \mathrm{M}(gy, gq, t_0) \\
&\quad = \mathrm{M}(g\varkappa, \varkappa, t_0) * \mathrm{M}(gy, y, t_0)
\end{aligned}$$

and $\quad \mathrm{M}(gy, y, \phi(t_0)) \geq \mathrm{M}(gy, y, t_0) * \mathrm{M}(g\varkappa, \varkappa, t_0)$.

Again using (8.1.1), we have

$$\begin{aligned}
&\mathrm{M}(g\varkappa, \varkappa, \phi^2(t_0)) \\
&\quad = \mathrm{M}(F(\varkappa, y), F(p, q), \phi^2(t_0)) \\
&\quad \geq \mathrm{M}(g\varkappa, gp, \phi(t_0)) * \mathrm{M}(gy, gq, \phi(t_0)) \\
&\quad = \mathrm{M}(g\varkappa, \varkappa, \phi(t_0)) * \mathrm{M}(gy, y, \phi(t_0)) \\
&\quad \geq [\mathrm{M}(g\varkappa, \varkappa, t_0)]^2 * [\mathrm{M}(gy, y, t_0)]^2,
\end{aligned}$$

and $\quad \mathrm{M}(gy, y, \phi^2(t_0)) \geq [\mathrm{M}(gy, y, t_0)]^2 * [\mathrm{M}(g\varkappa, \varkappa, t_0)]^2$.

Continuing likewise, for all $n \in \mathbb{N}$, we can get

$$\mathrm{M}(g\varkappa, \varkappa, \phi^n(t_0)) \geq [\mathrm{M}(g\varkappa, \varkappa, t_0)]^{2^{n-1}} * [\mathrm{M}(gy, y, t_0)]^{2^{n-1}},$$

$$\mathrm{M}(gy, y, \phi^n(t_0)) \geq [\mathrm{M}(gy, y, t_0)]^{2^{n-1}} * [\mathrm{M}(g\varkappa, \varkappa, t_0)]^{2^{n-1}}.$$

Then, we have

$$\begin{aligned}
&\mathrm{M}(g\varkappa, \varkappa, t) \\
&\quad \geq \mathrm{M}(g\varkappa, \varkappa, \textstyle\sum_{j=n_0}^{\infty} \phi^j(t_0)) \\
&\quad \geq \mathrm{M}(g\varkappa, \varkappa, \phi^{n_0}(t_0)) \\
&\quad \geq [\mathrm{M}(g\varkappa, \varkappa, t_0)]^{2^{n_0-1}} * [\mathrm{M}(gy, y, t_0)]^{2^{n_0-1}} \\
&\quad \geq \underbrace{(1-\varrho) * (1-\varrho) * \ldots (1-\varrho)}_{2^{n_0}} \geq (1-\sigma),
\end{aligned}$$

that is, for any $\sigma > 0$, we have $\mathrm{M}(g\varkappa, \varkappa, t) \geq (1-\sigma)$, for all $t > 0$.

Therefore, we get $g\varkappa = \varkappa$. Similarly, we can get $gy = y$.

Step 3. We now show that $\varkappa = y$.

Since $*$ is a t-norm of H-type, for any $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $t_0 > 0$ such that

$$\mathrm{M}(\varkappa, y, t_0) \geq (1-\varrho).$$

As $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{t} > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{t} > \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)$.

Using (8.1.1), we get

$$\mathrm{M}(\varkappa, y, \phi(\mathfrak{t}_0)) = \mathrm{M}(\mathrm{F}(p, q), \mathrm{F}(q, p), \phi(\mathfrak{t}_0))$$
$$\geq \mathrm{M}(gp, gq, \mathfrak{t}_0) * \mathrm{M}(gq, gp, \mathfrak{t}_0)$$
$$= \mathrm{M}(\varkappa, y, \mathfrak{t}_0) * \mathrm{M}(y, \varkappa, \mathfrak{t}_0).$$

Continuing likewise, for all $n \in \mathbb{N}$, we get

$$\mathrm{M}(\varkappa, y, \phi^n(\mathfrak{t}_0)) \geq [\mathrm{M}(\varkappa, y, \mathfrak{t}_0)]^{2^{n_0-1}} * [\mathrm{M}(y, \varkappa, \mathfrak{t}_0)]^{2^{n_0-1}},$$

then, we have

$$\mathrm{M}(\varkappa, y, \mathfrak{t}) \geq \mathrm{M}(\varkappa, y, \textstyle\sum_{k=n_0}^{\infty} \phi^k(\mathfrak{t}_0)) \geq \mathrm{M}(\varkappa, y, \phi^{n_0}(\mathfrak{t}_0))$$

$$\geq [\mathrm{M}(\varkappa, y, \mathfrak{t}_0)]^{2^{n_0-1}} * [\mathrm{M}(y, \varkappa, \mathfrak{t}_0)]^{2^{n_0-1}} \geq \underbrace{(1-\varrho) * (1-\varrho) * \dots (1-\varrho)}_{2^{n_0}} \geq (1-\sigma),$$

which implies that $\varkappa = y$. Therefore, F and g have a common fixed point $\varkappa$ in X.

Step 4. Finally, we show the uniqueness of $\varkappa$.

Let z be any point in X with $gz = z = \mathrm{F}(z, z)$.

Since $*$ is a t-norm of H-type, for any $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \dots * (1-\varrho)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $\mathfrak{t}_0 > 0$ such that

$$\mathrm{M}(\varkappa, z, \mathfrak{t}_0) \geq (1-\varrho).$$

As $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{t} > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{t} > \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)$.

Using (8.1.1), we have

$$\mathrm{M}(\varkappa, z, \phi(\mathfrak{t}_0)) = \mathrm{M}(\mathrm{F}(\varkappa, \varkappa), \mathrm{F}(z, z), \phi(\mathfrak{t}_0))$$
$$\geq \mathrm{M}(g\varkappa, gz, \mathfrak{t}_0) * \mathrm{M}(g\varkappa, gz, \mathfrak{t}_0)$$
$$= \mathrm{M}(\varkappa, z, \mathfrak{t}_0) * \mathrm{M}(\varkappa, z, \mathfrak{t}_0) = [\mathrm{M}(\varkappa, z, \mathfrak{t}_0)]^2.$$

Continuing likewise, for all $n \in \mathbb{N}$, we can obtain

$$\mathrm{M}(\varkappa, z, \phi^n(\mathfrak{t}_0)) \geq \left([\mathrm{M}(\varkappa, z, \mathfrak{t}_0)]^{2^{n-1}}\right)^2.$$

Then, we have

$$\mathrm{M}(\varkappa, z, \mathfrak{t}) \geq \mathrm{M}(\varkappa, z, \textstyle\sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)) \geq \mathrm{M}(\varkappa, z, \phi^{n_0}(\mathfrak{t}_0)) \geq \left([\mathrm{M}(\varkappa, z, \mathfrak{t}_0)]^{2^{n_0-1}}\right)^2$$

$$= [\mathrm{M}(\varkappa, z, \mathfrak{t}_0)]^{2^{n_0}} \geq \underbrace{(1-\varrho) * (1-\varrho) * \dots * (1-\varrho)}_{2^{n_0}} \geq (1-\sigma),$$

which implies that $\varkappa = z$. Hence, F and g have a unique common fixed point in X.

Remark 8.4.3. Theorem 8.4.1 generalizes Theorems 8.1.1 (Hu [146]) and 8.1.2 (Hu et al. [147]) for weakly compatible mappings along with (CLRg) property.

Theorem 8.4.1 does not require continuity hypothesis of any of the mappings involved and also relaxes the containment condition of the range subspace of the mapping Ꞙ into the range subspace of the mapping g. Further, the completeness of the space or the range subspaces has also been relaxed on using (CLRg) property.

Next, we give another generalization of Theorems 8.1.1 and 8.1.2 as follows:

Corollary 8.4.1. Let (Ӽ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being continuous t-norm of H-type. Let Ꞙ: Ӽ × Ӽ → Ӽ and g: Ӽ → Ӽ be two mappings and there exists $\phi \in \Phi_{\phi}$ satisfying (8.1.1). Suppose that the pair (Ꞙ, g) is weakly compatible and satisfies property (E.A.). If g(Ӽ) is closed subspace of Ӽ, then there exists a unique ϰ in Ӽ such that Ꞙ(ϰ, ϰ) = ϰ = gϰ.

Proof. As (Ꞙ, g) satisfy property (E.A.), there exist sequences $\{\varkappa_n\}$, $\{y_n\}$ in Ӽ such that $\{Ꞙ(\varkappa_n, y_n)\}$, $\{g\varkappa_n\}$ converges to ϰ and $\{Ꞙ(y_n, \varkappa_n)\}$, $\{gy_n\}$ converges to y for some ϰ, y in Ӽ, as n → ∞. Since g(Ӽ) is closed in Ӽ, so ϰ = gp, y = gq for some p, q in Ӽ. Consequently, the pair (Ꞙ, g) satisfies (CLRg) property. Now, by Theorem 8.4.1, Ꞙ and g have a unique common fixed point in Ӽ.

Remark 8.4.4. (i) The significance of (CLRg) property and property (E.A.) is that both the properties not only relaxes the continuity hypothesis of all the mappings involved but also relaxes the containment condition of the range subspace of the mapping into the range subspace of the other mapping.

(ii) It has been noticed that property (E.A.) replaces the completeness requirement of the space and range subspaces of the mappings with a more natural condition of the range subspaces to be closed whereas (CLRg) property ensures that one does not require even this condition also.

Next, we extend Theorem 8.4.1 for two pair of mappings sharing ($CLR_{ȘȚ}$) property and generalize Theorem 8.1.3 (Jain et al. [63]) as follows:

Theorem 8.4.2. Let (Ӽ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being continuous t-norm of H-type. Let Ⱥ, Ƀ: Ӽ × Ӽ → Ӽ and Ș, Ț: Ӽ → Ӽ be the mappings and there exists $\phi \in \Phi_{\phi}$ such that (8.1.2) holds. Suppose that the pairs (Ⱥ, Ș) and (Ƀ, Ț) share ($CLR_{ȘȚ}$) property and are w-compatible. Then, there exists a unique ɑ in Ӽ such that Ⱥ(ɑ, ɑ) = Șɑ = ɑ = Țɑ = Ƀ(ɑ, ɑ).

Proof. Since the pairs (Ⱥ, Ș) and (Ƀ, Ț) share ($CLR_{ȘȚ}$) property, there exist sequences $\{\varkappa_n\}$, $\{y_n\}$ and $\{ƥ_n\}$, $\{ɋ_n\}$ in Ӽ such that

$$\lim_{n\to\infty} Ⱥ(\varkappa_n, y_n) = \lim_{n\to\infty} Ƀ(ƥ_n, ɋ_n) = \lim_{n\to\infty} Țƥ_n = \lim_{n\to\infty} Ș\varkappa_n = ɑ,$$

$$\lim_{n\to\infty} Ꞁ(y_n, \varkappa_n) = \lim_{n\to\infty} Ƀ(ɋ_n, ꞵ_n) = \lim_{n\to\infty} Ţɋ_n = \lim_{n\to\infty} Şy_n = ɓ,$$

for some ɑ, ɓ ∈ Ş(Ӿ)∩Ţ(Ӿ).

Then, there exist some ⲣ, q, ɍ, s in Ӿ such that Şɍ = ɑ = Ţⲣ, Şs = ɓ = Ţq.

The proof is divided into following steps:

Step 1. We show that Ꞁ(ɑ, ɓ) = Şɑ, Ꞁ(ɓ, ɑ) = Şɓ and Ƀ(ɑ, ɓ) = Ţɑ, Ƀ(ɓ, ɑ) = Ţɓ.

Since $\phi \in \Phi_{\phi}$, we have $\phi(\mathfrak{t}) < \mathfrak{t}$ for all ƫ > 0. Then, using (8.1.2), for ƫ > 0, we have

$$Ϻ(Ꞁ(\varkappa_n, y_n), Ƀ(ⲣ, q), ƫ) \geq Ϻ(Ꞁ(\varkappa_n, y_n), Ƀ(ⲣ, q), \phi(ƫ))$$
$$\geq Ϻ(Ş\varkappa_n, Ţⲣ, ƫ) * Ϻ(Şy_n, Ţq, ƫ),$$

then, letting n → ∞ in the last inequality, for ƫ > 0, we obtain that

$$Ϻ(ɑ, Ƀ(ⲣ, q), ƫ) \geq Ϻ(ɑ, Ţⲣ, ƫ) * Ϻ(ɓ, Ţq, ƫ)$$
$$= Ϻ(ɑ, ɑ, ƫ) * Ϻ(ɓ, ɓ, ƫ)$$
$$= 1 * 1 = 1,$$

that is, Ϻ(ɑ, Ƀ(ⲣ, q), ƫ) = 1 and hence, Ƀ(ⲣ, q) = ɑ. Therefore, Ƀ(ⲣ, q) = ɑ = Ţⲣ. Similarly, we can show that Ƀ(q, ⲣ) = ɓ = Ţq.

Again, using (8.1.2), for ƫ > 0, we have

$$Ϻ(Ꞁ(ɍ, s), Ƀ(ꞵ_n, ɋ_n), ƫ) \geq Ϻ(Ꞁ(ɍ, s), Ƀ(ꞵ_n, ɋ_n), \phi(ƫ))$$
$$\geq Ϻ(Şɍ, Ţꞵ_n, ƫ) * Ϻ(Şs, Ţɋ_n, ƫ),$$

on letting n → ∞ in the last inequality, for ƫ > 0, we obtain that

Ϻ(Ꞁ(ɍ, s), ɑ, ƫ) ≥ Ϻ(Şɍ, ɑ, ƫ) ∗ Ϻ(Şs, ɓ, ƫ) = Ϻ(ɑ, ɑ, ƫ) ∗ Ϻ(ɓ, ɓ, ƫ) = 1 ∗ 1 = 1, so that, Ϻ(ɑ, Ꞁ(ɍ, s), ƫ) = 1 and hence, Ꞁ(ɍ, s) = ɑ. Therefore, Ꞁ(ɍ, s) = ɑ = Şr. Similarly, we can obtain that Ꞁ(s, ɍ) = ɓ = Şs.

Now, since the pair (Ƀ, Ţ) is w-compatible, so that Ƀ(ⲣ, q) = ɑ = Ţⲣ and Ƀ(q, ⲣ) = ɓ = Ţq implies that Ƀ(Ţⲣ, Ţq) = Ţ(Ƀ(ⲣ, q)) and Ƀ(Ţq, Ţⲣ) = Ţ(Ƀ(q, ⲣ)), that is, Ƀ(ɑ, ɓ) = Ţɑ and Ƀ(ɓ, ɑ) = Ţɓ.

Also, since the pair (Ꞁ, Ş) is w-compatible, so that Ꞁ(ɍ, s) = ɑ = Şɍ and Ꞁ(s, ɍ) = ɓ = Şs implies that Ꞁ(ɑ, ɓ) = Şɑ and Ꞁ(ɓ, ɑ) = Şɓ.

Step 2. We next show that Ꞁ(ɑ, ɓ) = Şɑ = ɑ = Ţɑ = Ƀ(ɑ, ɓ) and Ꞁ(ɓ, ɑ) = Şɓ = ɓ = Ţɓ = Ƀ(ɓ, ɑ).

Since ∗ is a t-norm of H-type, for $\sigma > 0$, there exists ϱ > 0 such that

$$\underbrace{(1-ϱ) * (1-ϱ) * \ldots * (1-ϱ)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $ƫ_0 > 0$, such that

$$Ϻ(ɑ, Ţɑ, ƫ_0) \geq (1-ϱ) \text{ and } Ϻ(ɓ, Ţɓ, ƫ_0) \geq (1-ϱ).$$

As $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{t} > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{t} > \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)$.

Using (8.1.2), we have

$$\begin{aligned} & M(a, Ta, \phi(\mathfrak{t}_0)) \\ &\quad = M(A(r, s), B(a, b), \phi(\mathfrak{t}_0)) \\ &\quad \geq M(Sr, Ta, \mathfrak{t}_0) * M(Ss, Tb, \mathfrak{t}_0) \\ &\quad = M(a, Ta, \mathfrak{t}_0) * M(b, Tb, \mathfrak{t}_0). \end{aligned}$$

Similarly, we can get

$$M(b, Tb, \phi(\mathfrak{t}_0)) \geq M(b, Tb, \mathfrak{t}_0) * M(a, Ta, \mathfrak{t}_0).$$

Now,

$$M(a, Ta, \phi^2(\mathfrak{t}_0)) = M(a, Ta, \phi(\phi(\mathfrak{t}_0))) \geq M(a, Ta, \phi(\mathfrak{t}_0)) * M(b, Tb, \phi(\mathfrak{t}_0))$$
$$\geq [M(a, Ta, \mathfrak{t}_0) * M(b, Tb, \mathfrak{t}_0)] * [M(a, Ta, \mathfrak{t}_0) * M(b, Tb, \mathfrak{t}_0)]$$
$$= [M(a, Ta, \mathfrak{t}_0)]^2 * [M(b, Tb, \mathfrak{t}_0)]^2.$$

Similarly, we can obtain that

$$M(b, Tb, \phi^2(\mathfrak{t}_0)) \geq [M(a, Ta, \mathfrak{t}_0)]^2 * [M(b, Tb, \mathfrak{t}_0)]^2.$$

Also, we have

$$M(a, Ta, \phi^3(\mathfrak{t}_0)) = M(a, Ta, \phi(\phi^2(\mathfrak{t}_0))) \geq M(a, Ta, \phi^2(\mathfrak{t}_0)) * M(b, Tb, \phi^2(\mathfrak{t}_0))$$
$$\geq [M(a, Ta, \mathfrak{t}_0)]^2 * [M(b, Tb, \mathfrak{t}_0)]^2 * [M(a, Ta, \mathfrak{t}_0)]^2 * [M(b, Tb, \mathfrak{t}_0)]^2$$
$$= [M(a, Ta, \mathfrak{t}_0)]^4 * [M(b, Tb, \mathfrak{t}_0)]^4.$$

Similarly, we can obtain that

$$M(b, Tb, \phi^3(\mathfrak{t}_0)) \geq [M(a, Ta, \mathfrak{t}_0)]^4 * [M(b, Tb, \mathfrak{t}_0)]^4.$$

Again, we have

$$M(a, Ta, \phi^4(\mathfrak{t}_0)) = M(a, Ta, \phi(\phi^3(\mathfrak{t}_0))) \geq M(a, Ta, \phi^3(\mathfrak{t}_0)) * M(b, Tb, \phi^3(\mathfrak{t}_0))$$
$$\geq [M(a, Ta, \mathfrak{t}_0)]^4 * [M(b, Tb, \mathfrak{t}_0)]^4 * [M(a, Ta, \mathfrak{t}_0)]^4 * [M(b, Tb, \mathfrak{t}_0)]^4$$
$$= [M(a, Ta, \mathfrak{t}_0)]^8 * [M(b, Tb, \mathfrak{t}_0)]^8.$$

Similarly, we can obtain

$$M(b, Tb, \phi^4(\mathfrak{t}_0)) \geq [M(a, Ta, \mathfrak{t}_0)]^8 * [M(b, Tb, \mathfrak{t}_0)]^8.$$

In general, for $n \geq 1$, we obtain that

$$M(a, Ta, \phi^n(\mathfrak{t}_0)) \geq [M(a, Ta, \mathfrak{t}_0)]^{2^{n-1}} * [M(b, Tb, \mathfrak{t}_0)]^{2^{n-1}}$$

and $$M(b, Tb, \phi^n(\mathfrak{t}_0)) \geq [M(a, Ta, \mathfrak{t}_0)]^{2^{n-1}} * [M(b, Tb, \mathfrak{t}_0)]^{2^{n-1}}.$$

Then, for $\mathfrak{t} > 0$, we have

$$\begin{aligned} & M(a, Ta, \mathfrak{t}) \\ &\quad \geq M(a, Ta, \textstyle\sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)) \\ &\quad \geq M(a, Ta, \phi^{n_0}(\mathfrak{t}_0)) \end{aligned}$$

$\geq [\mathrm{M}(\mathfrak{a}, \mathrm{Ţ}\mathfrak{a}, \mathfrak{t}_0)]^{2^{n_0-1}} * [\mathrm{M}(\mathfrak{b}, \mathrm{Ţ}\mathfrak{b}, \mathfrak{t}_0)]^{2^{n_0-1}}$

$\geq \underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{2^{n_0}} \geq (1-\sigma).$

Similarly, for $\mathfrak{t} > 0$, we can get

$$\mathrm{M}(\mathfrak{b}, \mathrm{Ţ}\mathfrak{b}, \mathfrak{t}) \geq (1-\sigma).$$

Therefore, for $\sigma > 0$, we have $\mathrm{M}(\mathfrak{a}, \mathrm{Ţ}\mathfrak{a}, \mathfrak{t}) \geq (1-\sigma)$ and $\mathrm{M}(\mathfrak{b}, \mathrm{Ţ}\mathfrak{b}, \mathfrak{t}) \geq (1-\sigma)$ for all $\mathfrak{t} > 0$, so that we have $\mathrm{Ţ}\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ţ}\mathfrak{b} = \mathfrak{b}$. Similarly, we can obtain $\mathrm{Ş}\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ş}\mathfrak{b} = \mathfrak{b}$. Therefore, we have

$$Ⱥ(\mathfrak{a}, \mathfrak{b}) = \mathrm{Ş}\mathfrak{a} = \mathfrak{a} = \mathrm{Ţ}\mathfrak{a} = Ƀ(\mathfrak{a}, \mathfrak{b}) \text{ and } Ⱥ(\mathfrak{b}, \mathfrak{a}) = \mathrm{Ş}\mathfrak{b} = \mathfrak{b} = \mathrm{Ţ}\mathfrak{b} = Ƀ(\mathfrak{b}, \mathfrak{a}).$$

Step 3. We assert that $\mathfrak{a} = \mathfrak{b}$.

Since $*$ is a t-norm of H-type, for $\sigma > 0$, there exists $\varrho > 0$ such that

$$\underbrace{(1-\varrho) * (1-\varrho) * \ldots * (1-\varrho)}_{i} \geq (1-\sigma), \text{ for all } i \in \mathbb{N}.$$

By (FM-6), there exists $\mathfrak{t}_0 > 0$, such that

$$\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0) \geq (1-\varrho).$$

As $\phi \in \Phi_\phi$, by (ϕ-3), for any $\mathfrak{t} > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{t} > \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)$.

Using (8.1.2), we have

$\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi(\mathfrak{t}_0))$

$= \mathrm{M}(Ⱥ(\mathfrak{a}, \mathfrak{b}), Ƀ(\mathfrak{b}, \mathfrak{a}), \phi(\mathfrak{t}_0))$

$\geq \mathrm{M}(\mathrm{Ş}\mathfrak{a}, \mathrm{Ţ}\mathfrak{b}, \mathfrak{t}_0) * \mathrm{M}(\mathrm{Ş}\mathfrak{b}, \mathrm{Ţ}\mathfrak{a}, \mathfrak{t}_0)$

$= \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0) * \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0) = [\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0)]^2.$

Also,

$\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi^2(\mathfrak{t}_0)) = \mathrm{M}(Ⱥ(\mathfrak{a}, \mathfrak{b}), Ƀ(\mathfrak{b}, \mathfrak{a}), \phi(\phi(\mathfrak{t}_0)))$

$\geq \mathrm{M}(\mathrm{Ş}\mathfrak{a}, \mathrm{Ţ}\mathfrak{b}, \phi(\mathfrak{t}_0)) * \mathrm{M}(\mathrm{Ş}\mathfrak{b}, \mathrm{Ţ}\mathfrak{a}, \phi(\mathfrak{t}_0)) = \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi(\mathfrak{t}_0)) * \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi(\mathfrak{t}_0))$

$= [\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi(\mathfrak{t}_0))]^2 \geq [\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0)]^4.$

In general, for $n \geq 1$, we have

$$\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi^n(\mathfrak{t}_0)) \geq [\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0)]^{2^n}.$$

Now, for $\sigma > 0$ and $\mathfrak{t} > 0$, we have

$\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}) \geq \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \sum_{j=n_0}^{\infty} \phi^j(\mathfrak{t}_0)) \geq \mathrm{M}(\mathfrak{a}, \mathfrak{b}, \phi^{n_0}(\mathfrak{t}_0))$

$$\geq [\mathrm{M}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t}_0)]^{2^{n_0}} \geq \underbrace{(1-\varrho) * (1-\varrho) * \ldots (1-\varrho)}_{2^{n_0}} \geq (1-\sigma),$$

which implies that $\mathfrak{a} = \mathfrak{b}$.

Step 4. Finally, we show the uniqueness of point $\mathfrak{a}$.

Let $\alpha \in \mathrm{X}$ such that $Ⱥ(\alpha, \alpha) = \mathrm{Ş}\alpha = \alpha = \mathrm{Ţ}\alpha = Ƀ(\alpha, \alpha)$. We claim that $\alpha = \mathfrak{a}$.

Since ∗ is a t-norm of H-type, for $\sigma > 0$, there exists ϱ > 0 such that

$$\underbrace{(1-ϱ) * (1-ϱ) * \ldots * (1-ϱ)}_{\mathrm{i}} \geq (1-\sigma), \text{ for all i} \in \mathbb{N}.$$

By (FϺ-6), there exists $ŧ_0 > 0$, such that

$$Ϻ(ɑ, \alpha, ŧ_0) \geq (1-ϱ).$$

As $\phi \in \Phi_\phi$, by (ϕ-3), for any ŧ > 0, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{j=n_0}^{\infty} \phi^j(ŧ_0)$.

By (8.1.2), we have

$Ϻ(ɑ, \alpha, \phi(ŧ_0)) = Ϻ(Ⱥ(ɑ, ɑ), Ƀ(\alpha, \alpha), \phi(ŧ_0)) \geq Ϻ(Şɑ, Ţ\alpha, ŧ_0) * Ϻ(Şɑ, Ţ\alpha, ŧ_0)$

$= Ϻ(ɑ, \alpha, ŧ_0) * Ϻ(ɑ, \alpha, ŧ_0) = [Ϻ(ɑ, \alpha, ŧ_0)]^2$.

In general, for n ≥ 1, we can obtain

$$Ϻ(ɑ, \alpha, \phi^n(ŧ_0)) \geq [Ϻ(ɑ, \alpha, ŧ_0)]^{2^n}.$$

Now, for $\sigma > 0$ and ŧ > 0, we have

$$Ϻ(ɑ, \alpha, ŧ) \geq Ϻ(ɑ, \alpha, \textstyle\sum_{j=n_0}^{\infty} \phi^j(ŧ_0)) \geq Ϻ(ɑ, \alpha, \phi^{n_0}(ŧ_0)) \geq [Ϻ(ɑ, \alpha, ŧ_0)]^{2^{n_0}}$$

$$\geq \underbrace{(1-ϱ) * (1-ϱ) * \ldots (1-ϱ)}_{2^{n_0}} \geq (1-\sigma),$$

so that, we can obtain ɑ = α.

This completes the proof of our result.

Remark 8.4.5. (i) On taking Ⱥ = Ƀ = Ꞙ and Ş = Ţ = ǥ in Theorem 8.4.2, we obtain Theorem 8.4.1.

(ii) Theorem 8.4.2 also generalizes Theorems 8.1.1 (Hu [146]) and 8.1.2 (Hu et al. [147]).

(iii) Theorem 8.4.2 generalizes Theorem 8.1.3 (Jain et al. [63]), since in Theorem 8.4.2, the completeness assumption of the space or the range subspaces has been relaxed entirely and further, the containment condition of range subspaces of mappings into the range subspaces of the other mappings has also been relaxed.

Theorem 8.4.3. Let (Ӽ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being continuous t-norm of H-type. Let Ⱥ, Ƀ: Ӽ × Ӽ → Ӽ and Ş, Ţ: Ӽ → Ӽ be the mappings and there exists $\phi \in \Phi_\phi$ such that (8.1.2) holds. Suppose that Ş(Ӽ) and Ţ(Ӽ) are closed subsets of Ӽ, the pairs (Ⱥ, Ş) and (Ƀ, Ţ) share common property (E.A.) and are w-compatible. Then, there exists a unique ɑ in Ӽ such that Ⱥ(ɑ, ɑ) = Şɑ = ɑ = Ţɑ = Ƀ(ɑ, ɑ).

Proof. As the pairs (Ⱥ, Ş) and (Ƀ, Ţ) share common property (E.A.), there exist sequences $\{ϰ_n\}$, $\{y_n\}$ and $\{ƥ_n\}$, $\{ɋ_n\}$ in Ӽ such that

$$\lim_{n\to\infty} Ⱥ(ϰ_n, y_n) = \lim_{n\to\infty} Ƀ(ƥ_n, ɋ_n) = \lim_{n\to\infty} Ţƥ_n = \lim_{n\to\infty} Şϰ_n = ɑ, \tag{8.4.1}$$

$$\lim_{n\to\infty} Ⱥ(y_n, ϰ_n) = \lim_{n\to\infty} Ƀ\left(ꝗ_n, ꞵ_n\right) = \lim_{n\to\infty} Ţꝗ_n = \lim_{n\to\infty} Şy_n = ƀ, \qquad (8.4.2)$$

for some ɑ, ƀ in Ꭓ.

Now, since Ş(Ꭓ) and Ţ(Ꭓ) are closed subsets of Ꭓ, then using (8.4.1), we have ɑ ∈ Ş(Ꭓ) and ɑ ∈ Ţ(Ꭓ), so that ɑ ∈ Ş(Ꭓ)∩Ţ(Ꭓ). Similarly, using (8.4.2), we can obtain ƀ ∈ Ş(Ꭓ)∩Ţ(Ꭓ). Hence, it follows that the pairs (Ⱥ, Ş) and (Ƀ, Ţ) shares the ($CLR_{ŞŢ}$) property. Now, applying Theorem 8.4.2, we can obtain the required result.

Remark 8.4.6. Theorem 8.4.3 also generalizes Theorems 8.1.1 − 8.1.3.

Theorem 8.4.4. Let (Ꭓ, Ϻ, ∗) be a FϺ-space with (FϺ-6), ∗ being a continuous t-norm of H-type. Let Ⱥ, Ƀ: Ꭓ × Ꭓ → Ꭓ and Ş, Ţ: Ꭓ → Ꭓ be the mappings and there exists $\phi \in \Phi_\phi$ satisfying (8.1.2). Suppose that

(a) the pair (Ⱥ, Ş) satisfies the ($CLR_Ş$) property, Ⱥ(Ꭓ × Ꭓ) ⊆ Ţ(Ꭓ), Ţ(Ꭓ) is a complete subspace of Ꭓ and $\{Ƀ(ꞵ_n, ꝗ_n)\}$, $\{Ƀ(ꝗ_n, ꞵ_n)\}$ converges for every sequences $\{ꞵ_n\}$, $\{ꝗ_n\}$ in Ꭓ, whenever $\{Ţꞵ_n\}$, $\{Ţꝗ_n\}$ converges;

or

(b) the pair (Ƀ, Ţ) satisfies the ($CLR_Ţ$) property, Ƀ(Ꭓ × Ꭓ) ⊆ Ş(Ꭓ), Ş(Ꭓ) is a complete subspace of Ꭓ and $\{Ⱥ(ϰ_n, y_n)\}$, $\{Ⱥ(y_n, ϰ_n)\}$ converges for every sequences $\{ϰ_n\}$, $\{y_n\}$ in Ꭓ, whenever $\{Şϰ_n\}$, $\{Şy_n\}$ converges.

Then, there exists a unique u in Ꭓ such that Ⱥ(u, u) = Şu = u = Ţu = Ƀ(u, u), if the pairs (Ⱥ, Ş) and (Ƀ, Ţ) are w-compatible.

Proof. W.L.O.G., let condition (a) holds, so that the pair (Ⱥ, Ş) satisfies ($CLR_Ş$) property. Then, there exist sequences $\{ϰ_n\}$ and $\{y_n\}$ in Ꭓ such that

$$\lim_{n\to\infty} Ⱥ(ϰ_n, y_n) = \lim_{n\to\infty} Şϰ_n = ɑ, \ \lim_{n\to\infty} Ⱥ(y_n, ϰ_n) = \lim_{n\to\infty} Şy_n = ƀ,$$

for some ɑ, ƀ ∈ Ş(Ꭓ).

By given condition, Ⱥ(Ꭓ × Ꭓ) ⊆ Ţ(Ꭓ) (with Ţ(Ꭓ) being complete), for each $\{ϰ_n\}$ and $\{y_n\}$ in Ꭓ, there correspond sequences $\{ꞵ_n\}$ and $\{ꝗ_n\}$ in Ꭓ such that $Ⱥ(ϰ_n, y_n) = Ţꞵ_n$ and $Ⱥ(y_n, ϰ_n) = Ţꝗ_n$. Therefore, $\lim_{n\to\infty} Ⱥ(ϰ_n, y_n) = \lim_{n\to\infty} Ţꞵ_n = ɑ$, $\lim_{n\to\infty} Ⱥ(y_n, ϰ_n) = \lim_{n\to\infty} Ţꝗ_n = ƀ$, so that ɑ, ƀ ∈ Ţ(Ꭓ). Thus, we conclude that ɑ, ƀ ∈ Ş(Ꭓ)∩Ţ(Ꭓ).

Next, we assert that $\{Ƀ(ꞵ_n, ꝗ_n)\}$ converges to ɑ and $\{Ƀ(ꝗ_n, ꞵ_n)\}$ converges to ƀ.

Now, since $\phi \in \Phi_\phi$, we have ϕ(ƚ) < ƚ for all ƚ > 0. Then, using (8.1.2), for ƚ > 0, we have

$Ϻ(Ⱥ(ϰ_n, y_n), Ƀ(ꞵ_n, ꝗ_n), ƚ)$

$\geq M(A(x_n, y_n), B(\beta_n, q_n), \phi(t)) \geq M(Sx_n, T\beta_n, t) * M(Sy_n, Tq_n, t)$,

then, letting $n \to \infty$ in the last inequality, we obtain that $\lim_{n\to\infty} M(a, B(\beta_n, q_n), t) = 1$, so that $\{B(\beta_n, q_n)\} \to a$ as $n \to \infty$. Similarly, we can obtain $\{B(q_n, \beta_n)\} \to b$ as $n \to \infty$. Hence, we have

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} B(\beta_n, q_n) = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} T\beta_n = a,$$

$$\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} B(q_n, \beta_n) = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tq_n = b,$$

for some $a, b \in S(X) \cap T(X)$. Therefore, the pairs (A, S) and (B, T) shares the (CLR_{ST}) property. Now, the proof is similar to the proof of Theorem 8.4.2.

8.5. APPLICATION TO METRIC SPACES

As application of the results proved in different sections of this chapter, we now formulate some corresponding results in metric spaces.

Theorem 8.5.1. Let (X, d) be a metric space and $A, B: X \times X \to X$ and $S, T: X \to X$ be four mappings and there exists some $k \in (0, 1)$ such that

$$\max\{d(A(x, y), B(u, v)), d(A(y, x), B(v, u))\} \leq \frac{k}{2}[d(Sx, Tu)+d(Sy, Tv)], \quad (8.5.1)$$

for all x, y, u, v in X. Also, suppose that $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$, the pairs (A, S) and (B, T) are weakly compatible, one of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete. Then, there exists a unique point α in X such that $A(\alpha, \alpha) = S\alpha = \alpha = T\alpha = B(\alpha, \alpha)$.

Proof. For all x, y in X and $t > 0$, define $M(x, y, t) = \frac{t}{t + d(x,y)}$ and $a * b = \min\{a, b\}$. Further, $M(x, y, t) \to 1$ as $t \to \infty$ for all x, y in X. Then, $(X, M, *)$ is a FM-space with (FM-6), where $*$ being the Hadžić type t-norm.

We now show that the inequality (8.5.1) implies (8.3.1) for $\phi(t) = kt$ with $t > 0$, $0 < k < 1$ and ω, γ being the identity mappings on their respective domains. If otherwise, from (8.3.1), for some $t > 0$ and x, y, u, v in X, we have

$$\min\left\{\frac{t}{t + \frac{1}{k} d(A(x,y), B(u,v))}, \frac{t}{t + \frac{1}{k} d(A(y,x), B(v,u))}\right\} < \min\left\{\frac{t}{t + d(Sx, Tu)}, \frac{t}{t + d(Sy, Tv)}\right\},$$

then, either

$$\frac{t}{t + \frac{1}{k} d(A(x,y), B(u,v))} < \min\left\{\frac{t}{t + d(Sx, Tu)}, \frac{t}{t + d(Sy, Tv)}\right\}, \quad (8.5.2)$$

or

$$\frac{t}{t + \frac{1}{k} d(A(y,x), B(v,u))} < \min\left\{\frac{t}{t + d(Sx, Tu)}, \frac{t}{t + d(Sy, Tv)}\right\}. \quad (8.5.3)$$

From (8.5.2), we get

$$\text{ƭ} + \frac{1}{k}\, d\big(A(\varkappa, y), B(u,v)\big) > \text{ƭ} + d(S\varkappa, Tu), \tag{8.5.4}$$

$$\text{ƭ} + \frac{1}{k}\, d\big(A(\varkappa, y), B(u,v)\big) > \text{ƭ} + d(Sy, Tv). \tag{8.5.5}$$

Combining (8.5.4) and (8.5.5), we get

$$d\big(A(\varkappa, y), B(u,v)\big) > \frac{k}{2}\,[d(S\varkappa, Tu) + d(Sy, Tv)]. \tag{8.5.6}$$

Similarly, by (8.5.3), we have

$$d\big(A(y, \varkappa), B(v, u)\big) > \frac{k}{2}\,[d(S\varkappa, Tu) + d(Sy, Tv)]. \tag{8.5.7}$$

Using (8.5.6) and (8.5.7), we get

$$\max\{d\big(A(\varkappa, y), B(u, v)\big), d\big(A(y, \varkappa), B(v, u)\big)\} > \frac{k}{2}\,[d(S\varkappa, Tu) + d(Sy, Tv)],$$

a contradiction to (8.5.1). Then, the result holds immediately by applying Theorem 8.3.1.

Theorem 8.5.2. Theorem 8.5.1 remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypotheses) of the following properties:

(i) Compatibility;

(ii) COM(A);

(iii) COM(P);

(iv) COM(B);

(v) COM(C);

(vi) COM(A_F);

(vii) COM(A_g);

(viii) Commuting;

(ix) WC;

(x) R-WC;

(xi) R-WC(A_F);

(xii) R-WC(A_g);

(xiii) R-WC(P).

Proof. The proof follows immediately by using the relationship between weakly compatible mappings and the variants of weakly commuting and compatible mappings.

Theorem 8.5.3. Let (X, d) be a metric space and $A, B: X \times X \to X$ and $S, T: X \to X$ be four mappings and there exists some $k \in (0, 1)$ such that

$$d(A(\varkappa, y), B(u, v)) \leq \frac{k}{2}\,[d(S\varkappa, Tu) + d(Sy, Tv)], \tag{8.5.8}$$

for all $\varkappa$, y, u, v in X. Also, suppose that the pairs (A, S) and (B, T) share (CLR_{ST}) property and are w-compatible. Then, there exists a unique point α in X such that $A(\alpha, \alpha) = S\alpha = \alpha = T\alpha = B(\alpha, \alpha)$.

Proof. For all $\varkappa$, y in X and $t > 0$, define $M(\varkappa, y, t) = \frac{t}{t + d(\varkappa,y)}$ and $a * b = \min\{a, b\}$. Further, $M(\varkappa, y, t) \to 1$ as $t \to \infty$ for all $\varkappa$, y in X. Then, (X, M, $*$) is a FM-space with (FM-6), where $*$ being the Hadžić type t-norm. We next prove that the inequality (8.5.8) implies (8.1.2) for $\phi(t) = kt$ with $t > 0$ and $0 < k < 1$. If otherwise, from (8.1.2), for some $t > 0$ and $\varkappa$, y, u, v $\in X$, we have

$$\frac{t}{t + \frac{1}{k} d(A(\varkappa,y), B(u,v))} < \min\left\{\frac{t}{t + d(S\varkappa, Tu)}, \frac{t}{t + d(Sy, Tv)}\right\},$$

then, we have

$$\frac{t}{t + \frac{1}{k} d(A(\varkappa,y), B(u,v))} < \min\left\{\frac{t}{t + d(S\varkappa, Tu)}, \frac{t}{t + d(Sy, Tv)}\right\},$$

which implies that

$$t + \frac{1}{k} d(A(\varkappa, y), B(u,v)) > t + d(S\varkappa, Tu), \qquad (8.5.9)$$

$$t + \frac{1}{k} d(A(\varkappa, y), B(u,v)) > t + d(Sy, Tv). \qquad (8.5.10)$$

Combining (8.5.9) and (8.5.10), we get

$$d(A(\varkappa, y), B(u,v)) > \frac{k}{2}[d(S\varkappa, Tu) + d(Sy, Tv)],$$

a contradiction to (8.5.8).

Then, the result holds immediately by applying Theorem 8.4.2.

CONCLUSION

The outcome of the present work is in accordance with the objectives proposed in section 1.7. Present work extends various notions present in the literature. Further, the results obtained generalize and extend a number of existing works. We shall discuss the conclusion of the presented work as follows:

- New (φ, ψ) – contractive conditions are introduced in POMS and POPMS. Using theses notions some results in coupled fixed point theory are formulated that generalize and extend the recent results of Berinde [149, 150]. Further, the obtained results weaken the results of Bhaskar and Lakshmikantham [55], Luong and Thuan [67] and Alotaibi and Alsulami [68].
- The notion of generalized symmetric g-Meir-Keeler type contraction has been introduced and utilized to obtain coupled common fixed points in the setup of POMS. This notion extends the notion of generalized symmetric Meir-Keeler contractions due to Berinde and Pacurar [155].
- The notions of (α, ψ) - weak contraction conditions in POMS have been introduced and utilized for establishing coupled common and coupled fixed point results. Our work improves and extends the main result of Karapinar and Agarwal [158] to the pair of compatible mappings and generalizes the recent results of Jain et al. [159], Berinde [149] and weakens the contractions involved in the works of Bhaskar and Lakshmikantham [55] and Mursaleen et al. [157]. Further, a new concept of α - regular spaces has also been designed.
- New contractions in the setup of G-metric spaces have been framed and utilized to formulate coupled common fixed point results. Recent works of Choudhury and Maity [103], Nashine [161], Karapinar et al. [162], Mohiuddine and Alotaibi [163] and Jain and Tas [164] have been generalized.
- As applications of the obtained results some results of integral type have been obtained. Further, applications to the solutions of integral equations have also been achieved.
- The technique introduced by Sintunavarat et al. [166] which was utilized by Hussain et al. [167] to compute coupled coincidence points has been improved. Utilizing the new improved technique, the recent results of

Hussain et al. [167], Choudhury et al. [56], Alsulami [168], Choudhury et al. [119] have been improved.

- The errors and omissions in the recent papers of Alotaibi and Alsulami [68], Turkoglu and Sangurlu [169], Zhu et al. [120], Singh and Jain [170] are rectified.
- Fixed point results for generalized weak ($\psi > \phi$) – contraction mappings are established in the setup of POMS. Further, some corresponding results in coupled fixed point theory are also formulated. The obtained results generalize the recent works of Ran and Reurings [40], Nieto and López [41], Harjani and Sadarangani [47, 48], Amini-Harandi and Emami [53], Ćirić et al. [52], Bhaskar and Lakshmikantham [55], Harjani et al. [58], Choudhury et al. [56], Berinde [149], Rasouli and Baharampour [70], Jain et al. [159].
- Coupled coincidence point and coupled common fixed point results for the pair of mappings lacking MgMP under a new generalized nonlinear contractive condition are obtained. The obtained results generalize the results of Bhaskar and Lakshmikantham [55], Harjani et al. [58], Rasouli and Bahrampour [70], Choudhury et al. [56], Luong and Thuan [69], Karapinar et al. [57] and Chandok and Tas [174].
- Relationship among the variants of weakly commuting mappings is obtained in context of coupled fixed point theory. Further, relationship among the variants of compatible mappings is also attained.
- The notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property are designed in the context of coupled fixed point theory. Utilizing these notions some results are obtained that generalize certain existing results. In particular, the results of Hu [146], Hu et al. [147] and Jain et al. [63] are generalized in FM-spaces.
- Metrical version of some results proved in FM-spaces has also been obtained.

SCOPE FOR FURTHER WORK

The work done in the thesis fulfils the objectives of the present study. However, while achieving the proposed objectives, some new avenues of research get opened for further investigation as mentioned below:

- The work for variants of weakly commuting mappings and variants of compatible mappings in coupled fixed theory is in initial stage, many interesting results can be obtained using these variants under different contractions.
- In the context of coupled fixed point theory, the work comprising the notions of property (E.A.), (CLRg) property, common property (E.A.) and (CLR_{ST}) property is in its inceptive stage, many important results can be attained utilizing these notions.
- The notions of generalized symmetric g-Meir-Keeler type contraction and (α, ψ) - weak contraction conditions can be defined in G-metric spaces and can be used to obtain the existence and uniqueness of coupled common fixed points for the pair of mappings.
- The new improved technique formulated in Chapter – VI can be used to generalize more results present in various spaces.
- Fixed points for generalized weak $(\psi > \phi)$ – contraction mappings can be obtained in partial as well as in G-metric spaces.

As this theory is developing enormously day-by-day, more interesting outcomes can be drawn in it.

REFERENCES

[1] S. G. Matthews, “Partial metric topology”, Proc. 8th Summer Conference on General Topology and applications, in: *Ann. New York Acad. Sci.,* Vol. 728, 1994, pp. 183-197.

[2] S. Gahler, “2-metrische Raume und ihre topologische Struktur”, *Math. Nachr*, 26 (1963), pp. 115-148.

[3] C. R. Hsiao, “A Property of contractive type mappings in 2-metric spaces”, *Jnanabha*, 16 (1986), pp. 223-239.

[4] K. S. Ha, Y. J. Cho and A. White, “Strictly convex and strictly 2-convex 2-normed spaces”, *Mathematica Japonica*, 33 (3) (1988), pp. 375-384.

[5] B.C. Dhage, A study of some fixed point theorems”, *Ph. D. Thesis*, Marathwada University, Aurangabad, India (1984).

[6] Z. Mustafa and B. Sims, “Some remarks concerning D-metric spaces”, *Proceedings of International Conference on Fixed Point Theory and Applications*, Valencia (Spain), (July, 2003), pp. 189-198.

[7] Z. Mustafa and B. Sims, “A new approach to generalized metric spaces”, *Journal of Nonlinear and Convex Analysis*, 7(2) (2006), pp. 289-297.

[8] Z. Mustafa, H. Obiedat and Awawdeh, “Some fixed point theorem for mapping on complete G-metric spaces”, *Fixed Point Theory Appl.*, Vol 2008 (2008), Article ID 189870, 12 pages.

[9] K. Menger, “Statistical metrics”, *Proc. Nat. Acad. Sci. USA*, 28 (1942), pp. 535–537.

[10] B. Schweizer and A. Sklar, “Statistical Metric Spaces”, *Pacific J. Math.*, 10(1) (1960), pp. 313-334.

[11] B. Schweizer and A. Sklar, “Probabilistic Metric Spaces”, *North Holland Series in Probability and Applied Math.* 5, 1983.

[12] D. O'. Regan and R. Saadati, “Nonlinear contraction theorems in probabilistic spaces”, *Appl. Math. Comput.*, 195 (2008), pp. 86-93.

[13] V. M. Sehgal, “Some fixed point theorems in function analysis and probability”, *Ph.D dissertation*, Wayne State Univ. Michigan (1966).

[14] O. Hadžić and E. Pap, “Fixed Point Theory in Probabilistic Metric Spaces”, Vol. 536 of *Mathematics and its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[15] C. Zhou, S. Wang, Lj. Ćirić and S. M. Alsulami, "Generalized probabilistic metric spaces and fixed point theorems", *Fixed Point Theory Appl.*, 2014**, 2014:**91.

[16] L. A. Zadeh, "Fuzzy Sets", *Information and Control,* 89(1965), pp. 338-353.

[17] Z. K. Deng, "Fuzzy pseudo metric spaces", *J. Math. Anal. Appl.*, 86 (1982), pp. 74-95.

[18] M. A. Erceg, "Metric spaces in fuzzy set theory", *J. Math. Anal. Appl.*, 69 (1979), pp. 205-230.

[19] O. Kaleva and S. Seikkala, "On fuzzy metric spaces", *Fuzzy Sets and Systems*, 12(1984), pp. 215-229.

[20] I. Kramosil and J. Michalek, "Fuzzy Metric and Statistical metric Spaces", *Kybernetica*, 11(1975), pp. 326-334.

[21] A. George and P. Veeramani, "On some results in fuzzy metric spaces", *Fuzzy Sets and Systems*, 64(3) (1994), pp. 395-399.

[22] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces", *Fuzzy Sets and Systems*, 90 (1997), pp. 365-368.

[23] V. Gregori and A. Sapena, "On fixed-point theorems in fuzzy metric spaces", *Fuzzy Sets and Systems*, 125 (2002), pp. 245-252.

[24] P. P. Murthy, U. Mishra, Rashmi, C. Vetro, "Generalized (φ, ψ)-weak contractions involving (f, g)-reciprocally continuous maps in fuzzy metric spaces", *Ann. Fuzzy Math. Inform.*, 5(1) (2013), pp. 45-57.

[25] B. Singh and M. S. Chauhan, "Common fixed points of compatible maps in fuzzy metric spaces", *Fuzzy Sets and Systems*, 115 (2000), pp. 471-475.

[26] M. Jain, S. Kumar and M. Kumar, "Common fixed point theorem in fuzzy metric spaces with an application in the product space", *J. Ana. Num. Theor.*, 4(2) (2016), pp. 139-143.

[27] B. S. Choudhury, K. Das and P. Das, "Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces", *Fuzzy Sets and Systems,* 222 (2013), pp. 84-97.

[28] B. S. Choudhury and P. Das, "A new contraction mapping principle in partially ordered fuzzy metric spaces", *Ann. Fuzzy Math. Inform.*, 8(6) (2014), pp. 889 - 901.

[29] M. Aamri and D. El-Moutawakil, “Some new common fixed point theorems under strict contractive conditions”, *J. Math. Anal. Appl.*, 27 (2002), pp. 181-188.

[30] W. Sintunavarat and P. Kumam, “Common fixed point theorems for a pair of weakly compatible mappings in Fuzzy Metric Spaces”, *J. Appl. Math.*, Vol. 2011, Article ID 637958, 14 pages. doi: 10.1155/2011/637958.

[31] L. E. J. Brouwer, “Über Abbildungen von Mannigfaltigkeiten”, *Math. Ann.*, 71(1912), pp. 97–115.

[32] J. Schauder, “Der Fixpunktsatz in Funktionalräumen”, *Studia Math.*, 2 (1930), pp. 171–180.

[33] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations integrals”, *Fund. Math.*, 3(1922), pp. 133–181.

[34] D. W. Boyd and J. S. W. Wong, “On nonlinear contractions”, *Proc. Amer. Math. Soc.*, 20 (1969), pp. 458-464.

[35] V. Berinde, “Generalized contractions and applications (in Romanian)”, *Editura Cub Press* 22, Baia Mare, 1997.

[36] I. A. Rus, “Generalized contractions”, *Seminar on Fixed Point Theory*, 3, 1983, pp. 1-130.

[37] I. A. Rus, “Generalized contractions and applications”, *Cluj University Press*, Cluj-Napoca, 2001.

[38] A. Meir and E. Keeler, “A theorem on contraction mappings”, *J. Math. Anal. Appl.*, 28 (1969), pp. 326-329.

[39] M. A. Geraghty, “On contractive mappings”, *Proc. Amer. Math. Soc.*, 40(2) (1973), pp. 604-608.

[40] A. C. M. Ran and M. C. B. Reurings, “A fixed point theorem in partially ordered sets and some applications to matrix equations”, *Proc. Amer. Math. Soc.*, 132(5) (2004), pp. 1435–1443.

[41] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations”, *Order*, 22(3) (2005), pp. 223–239.

[42] J. J. Nieto and R. Rodríguez-López, “Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations”, *Acta Math. Sin. (Engl. Ser.)*, 23(12) (2007), pp. 2205-2212.

[43] M. S. Khan, M. Swaleh and S. Sessa, "Fixed point theorems by altering distances between the points", *Bull. Aust. Math. Soc.*, 30(1) (1984), pp. 1-9.

[44] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces", In: I. Gohberg, Y. Lyubich (eds) New Results in Operator Theory and its Applications, *Operator Theory: Advances and Application*, 98 (1997), 7-22, Birkhäuser, Basel.

[45] B. E. Rhoades, "Contractive definitions", *Nonlinear Analysis*, (e.d. T.M. Rassias) World Scientific Publishing Company, New Jerscy (1998), pp. 513-526.

[46] P. N. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces", *Fixed Point Theory Appl.*, 2008 Article ID 406368.

[47] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contraction mappings in partially ordered sets", *Nonlinear Anal.*, 71 (2009), pp. 3403-3410.

[48] J. Harjani and K. Sadarangani, "Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations", *Nonlinear Anal.*, 72 (2010), pp. 1188-1197.

[49] G. Jungck, "Commuting mappings and fixed points", *Amer. Math. Monthly*, 83 (1976), pp. 261–263.

[50] G. Jungck and B. E. Rhoades, "Fixed points for set valued functions without continuity", *Indian J. Pure Appl. Math.*, 29(3) (1998), pp. 227–238.

[51] R. P. Agarwal, M. A. El-Gebeily and D.O.'Regan, "Generalized contractions in partially ordered metric spaces", *Appl. Anal.*, 87(1) (2008), pp. 109–116.

[52] Lj. Ćirić, N. Cakić, M. Rajović and J. S. Ume, "Monotone Generalized Nonlinear Contractions in Partially Ordered Metric Spaces", *Fixed Point Theory Appl.*, Vol. 2008, Article ID 131294, 11 pages, doi:10.1155/2008/131294.

[53] A. Amini-Harandi and H. Emami, "A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations", *Nonlinear Anal.*, 72(2010), pp. 2238-2242.

[54] D. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications", *Nonlinear Anal.*, 11(1987), pp. 623-632.

[55] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications", *Nonlinear Anal.*, 65(7) (2006), pp. 1379–1393.

[56] B. S. Choudhury, N. Metiya and A. Kundu, "Coupled coincidence point theorems in ordered metric spaces", *Ann. Univ. Ferrara*, 57 (2011), pp. 1-16.

[57] E. Karapinar, N. V. Luong and N. X. Thuan, "Coupled coincidence points for mixed monotone operators in partially ordered metric spaces", *Arab J. Math.*, 1, pp 329-339. doi 10.1007/s40065-012-0027-0

[58] J. Harjani, B. López and K. Sadarangani, "Fixed point theorems for mixed monotone operators and applications to integral equations", *Nonlinear Anal.*, 74(2010), pp. 1749–1760.

[59] V. Lakshmikantham and Lj. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces", *Nonlinear Anal.*, 70 (2009), pp. 4341–4349.

[60] B. S. Choudhury and A. Kundu, "A coupled coincidence point result in partially ordered metric spaces for compatible mappings", *Nonlinear Anal.*, 73 (2010), pp. 2524-2531.

[61] M. Abbas, M. A. Khan and S. Radenović, "Common coupled fixed point theorem in cone metric space for *w*-compatible mappings", *Appl. Math. Comput.*, 217 (2010), pp. 195-202.

[62] A. F. Roldán-López-de-Hierro and W. Sintunavarat, "Common fixed point theorems in fuzzy metric spaces using CLRg property", *Fuzzy Sets and Systems*, 282 (2016), pp. 131-142.

[63] M. Jain, S. Kumar and R. Chugh, "Coupled fixed point theorems for weak compatible mappings in fuzzy metric spaces", *Ann. Fuzzy Math. Inform.*, 5(2) (2013), pp. 321-336.

[64] R. P. Agarwal, W. Sintunavarat and P. Kumam, "Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property", *Fixed Point Theory Appl.*, 2013, **2013**:22.

[65] H. K. Nashine, Z. Kadelburg and S. Radenović, "Coupled common fixed point theorems for w^*-compatible mappings in ordered cone metric spaces", *Appl. Math. Comput.*, 218 (2012), pp. 5422-5432.

[66] W. Sintunavarat, A. Petruşel and P. Kumam, "Common coupled fixed point theorems for w^*-compatible mappings without mixed monotone property", *Rend. Circ. Mat. Palermo*, 61 (2012), pp. 361-383. doi:10.1007/s12215-012-0096-0.

[67] N. V. Luong and N. X. Thuan, "Coupled fixed points in partially ordered metric spaces and application", *Nonlinear Anal.*, 74 (2011), pp. 983-992.

[68] A. Alotaibi and S. M. Alsulami, "Coupled coincidence points for monotone operators in partially ordered metric spaces", *Fixed Point Theory Appl.*, 2011, **2011**:44.

[69] N. V. Luong and N. X. Thuan, "Coupled fixed point theorems in partially ordered metric spaces", *Bull. Math. Anal. Appl.*, 2 (2010), pp. 16-24.

[70] S. H. Rasouli and M. Bahrampour, "A remark on the coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces", *J. Math. Comput. Sci.,* 3(2) (2011), pp. 246-261.

[71] O. Valero, "On Banach fixed point theorems for partial metric spaces", *Appl. Gen. Topol.*, 6 (2005), pp. 229-240.

[72] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces", *Rend. Istit. Mat. Univ. Trieste*, 36 (2004), pp. 17-26.

[73] I. Altun, F. Sola and H. Simsek, "Generalized contractions on partial metric spaces", *Topology Appl.*, 157 (18) (2010), pp. 2778-2785.

[74] D. Illić, V. Pavlović and V. Rakocević, "Some new extensions of Banach's contraction principle to partial metric spaces", *Appl. Math. Lett.*, 24 (2011), pp. 1326-1330.

[75] R. P. Agarwal, M. A. Alghamdi and N. Shahzad, "Fixed point theory for cyclic contractions in partial metric spaces", *Fixed Point Theory Appl.*, 2012, **2012**:40.

[76] R. Pant, R. Shukla, H. K. Nashine and R. Panicker, "Some new fixed point theorems in partial metric spaces with applications", *Journal of Function Spaces,* Vol. 2017, Article ID 1072750, 13 pages, 2017.

[77] K. P. Chi, E. Karapinar and T. D. Thanh, "A generalized contraction principle in partial metric spaces", *Math. Comput. Modelling,* 55 (5-6) (2012), pp. 1673 – 1681.

[78] S. Romaguera, "Fixed point theorems for generalized contractions on partial metric spaces", *Topology Appl.*, 159 (1) (2012), pp. 194-199.

[79] Lj. Ćirić, B. Samet, H. Aydi and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application", *Appl. Math. Comput.,* 218 (2011), pp. 2398-2406.

[80] I. Altun and S. Romaguera, “Characterizations of partial metric completeness in terms of weakly contractive mappings having fixed point”, *Appl. Anal. Discrete Math.*, 6(2) (2012), pp. 247-256.

[81] W. Shatanawi, B. Samet and M. Abbas, “Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces”, *Math. Comput. Modelling*, 55 (2012), pp. 680-687.

[82] H. Aydi, “Coupled fixed point results in ordered partial metric spaces”, *Selcuk J. Appl. Math.*, 13(1) (2012), pp. 23-33.

[83] H. Aydi, “Some fixed point results in ordered partial metric spaces”, *J. Nonlinear Sci. Appl.*, 4(3) (2011), pp. 210-217.

[84] T. Abdeljawad, E. Karapinar and K. Tas, “Existence and uniqueness of a common fixed point on partial metric spaces”, *Appl. Math. Lett.*, 24(11) (2011), pp. 1900-1904.

[85] T. Abdeljawad, “Fixed points for generalized weakly contractive mappings in partial metric spaces”, *Math. Comput. Modelling*, 54 (2011), pp. 2923-2927.

[86] M. Abbas and T. Nazir, “Fixed point of generalized weakly contractive mappings in ordered partial metric spaces”, *Fixed Point Theory Appl.*, 2012, **2012**:1.

[87] H. Aydi, “Some coupled fixed point results on partial metric spaces”, *Int. J. Math. Math. Sci.*, Vol. 2011, Article ID 647091, 11 pages.

[88] F. Sabetghadam, H. P. Masiha and A. H. Sanatpour, “Some coupled fixed point theorems in cone metric spaces”, *Fixed Point Theory Appl.*, Article ID 125426, 8 pages, 2009.

[89] S. Alsulami, N. Hussain and A. Alotaibi, “Coupled fixed and coincidence points for monotone operators in partial metric spaces”, *Fixed Point Theory Appl.*, 2012, **2012**:173.

[90] Z. Mustafa, “A new structure for generalized metric spaces with applications to fixed point theory”, *Ph. D. Thesis*, University of Newcastle, Newcastle, UK, 2005.

[91] W. Shatanawi, “Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces”, *Fixed Point Theory Appl.*, Vol. 2010, Article ID 181650, 9 pages.

[92] Z. Mustafa, H. Aydi and E. Karapinar, “Generalized Meir-Keeler type contractions on G-metric spaces”, *Appl. Math. Comput.*, 219 (21) (2013), pp. 10441-10447.

[93] N. Bilgili and E. Karapinar, "Cyclic contractions via auxiliary functions on G-metric spaces", *Fixed Point Theory Appl.*, 2013, **2013**:49.

[94] Y. U. Gaba, "Fixed point theorems in G-metric spaces", *J. Math. Anal. Appl.*, 455(1) (2017), pp. 528-537.

[95] R. Chugh, T. Kadian, A. Rani and B. E. Rhoades, "Property P in G-metric spaces", *Fixed Point Theory Appl.*, Vol. 2010, Article ID 401684, 12 pages.

[96] M. Abbas and B. E. Rhoades, "Common fixed point results for non-commuting mappings without continuity in generalized metric spaces", *Appl. Math. Lett.*, 215 (2009), pp. 262-269.

[97] W. Shatanawi, M. Abbas and H. Aydi, "On weakly C-contractive mappings in generalized metric spaces" (*submitted for publication*).

[98] H. Aydi, W. Shatanawi and C. Vetro, "On generalized weakly G-contraction mapping in G-metric spaces", *Comput. Math. Appl.*, 62 (2011), pp. 4222-4229.

[99] H.K. Nashine, Z. Kadelburg and S. Radenovic, "Coincidence and Fixed Point Results under Generalized Weakly Contractive Condition in Partially Ordered G-metric Spaces", *Filomat*, 27(7) (2013), pp. 1333–1343.

[100] M. Abbas, T. Nazir and S. Radenović, "Common fixed point of generalized weakly contractive maps in partially ordered G-metric spaces", *Appl. Math. Comput.*, 218 (18) (2012), pp. 9383-9395.

[101] A. Erduran and I. Altun, "Fixed point theorems for weakly contractive mapping on G-metric spaces and a homotopy result", *Int. J. Nonlinear Anal. Appl.*, 3 (1) (2012), pp. 1-8.

[102] Z. Mustafa, V. Parvaneh, M. Abbas and J. R. Roshan, "Some coincidence point results for generalized (ψ, φ)-weakly contractive mappings in ordered G-metric spaces", *Fixed Point Theory Appl.*, 2013, **2013**:326.

[103] B. S. Choudhury and P. Maity, "Coupled fixed point results in generalized metric spaces", *Math. Comput. Modelling*, 54 (2011), pp. 73-79.

[104] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, "Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces", *Math. Comput. Modelling*, 54 (2011), pp. 2443-2450.

[105] Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet and W. Shatanawi, "Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type", *Fixed Point Theory Appl.*, 2012, **2012**:8.

[106] V. M. Sehgal and A. T. Bharucha-Reid, "Fixed points of contraction mappings on probabilistic metric spaces", *Mathamtical Systems Theory*, 6(2) (1972), pp. 97-102.

[107] H. Sherwood, "Complete probabilistic metric spaces", *Z Wahrscheinlichkeitstheor Verw Geb*, 20 (1971), pp. 117-128.

[108] G. Bocsan, "On some fixed point theorems in probabilistic metric spaces", *Math. Balkanica (N.S.)*, 4 (1974), pp. 67-70.

[109] G. L. Cain Jr and R. H. Kasriel, "Fixed and periodic points of local contraction mappings on probabilistic metric spaces", *Mathematical Systems Theory*, 9 (1976), pp 289-297.

[110] V. I. Istrătescu and E. Roventa, "On fixed point theorems for mappings on probabilistic locally convex spaces", *Seminarul de teoria functiilor si matematici aplicate, A. Spatti metrice probabiliste*, 19 (1974).

[111] V. I. Istrătescu and I. Săcuiu, "Fixed point theorem for contraction mappings on probabilistic metric spaces", *Rev. Roumaine Math. Pures Appl.*, 18 (1973), pp. 1375–1380.

[112] Lj. Ćirić, "Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces", *Nonlinear Anal.*, 72 (2010), pp. 2009 – 2018.

[113] J. Jachymski, "On probabilistic φ - contractions on Menger spaces", *Nonlinear Anal.*, 73 (2010), pp. 2199 – 2203.

[114] J. X. Fang, "On φ – contractions in probabilistic and fuzzy metric spaces", *Fuzzy Sets and Systems*, 267 (2015), pp. 86-99.

[115] Lj. Ćirić, D. Mihet and R. Saadati, "Monotone generalized contractions in partially ordered probabilistic metric spaces", *Topology Appl.*, 156 (2009), pp. 2838-2844.

[116] Lj. Ćirić, R. P. Agarwal and B. Samet, "Mixed monotone-generalized contractions in partially ordered probabilistic metric spaces", *Fixed Point Theory Appl.*, 2011, **2011**:56.

[117] S. Wang, G. Gu and Y. J. Cho, "Coupled Fixed Point Theorems in Partially Ordered Menger Spaces", *Bull. Malays. Math. Sci. Soc. (2),* 37(2) (2014), pp. 531–542.

[118] D. Đorić, "Nonlinear coupled coincidence and coupled fixed point theorems for not necessary commutative contractive mappings in partially ordered

probabilistic metric spaces", *Appl. Math. Comput.*, 219 (11) (2013), pp. 5926-5935.

[119] B. S. Choudhury, P. Das and P. Saha, "Coupled coincidence point results for probabilistic φ-contractions", *Cogent Mathematics*, 3(1) (2016), 1204069.

[120] C. Zhu, X. Mu and Z. Wu, "Common coupled fixed point results for probabilistic φ-contractions in Menger PGM-spaces", *J. Nonlinear Sci. Appl.*, 8 (2015), pp. 1166 – 1175.

[121] M. Grabiec, "Fixed points in fuzzy metric spaces", *Fuzzy Sets and Systems*, 27 (3) (1988), pp. 385-389.

[122] J. X. Fang, "On fixed point theorems in fuzzy metric spaces", *Fuzzy Sets and Systems,* 46 (1992), pp. 107-113.

[123] M. Edelstein, "On fixed points and periodic points under contractive mappings", *J. Lond. Math. Soc.*, 37 (1962), pp. 74-79.

[124] I. Istrătescu, "A fixed point theorem for mappings with a probabilistic contractive iterate", *Rev. Roumaine Math. Pures Appl.*, 26 (1981), pp. 431-435.

[125] P. V. Subrahmanyam, "A common fixed point theorem in fuzzy metric spaces", *Inform. Sci.,* 83 (1995), pp. 109-112.

[126] G. Jungck, "Compatible mappings and common fixed points", *Int. J. Math. Math. Sci.*, 9 (1986), pp. 771-774.

[127] S. N. Mishra, N. Sharma and S. L. Singh, "Common fixed points of maps on fuzzy metric spaces", *Int. J. Math. Math. Sci.*, 17(2) (1994), pp. 253-258.

[128] R. Vasuki, "Common fixed points for R-weakly commuting maps in fuzzy metric spaces", *Indian J. Pure App. Math.*, 30(4) (1999), pp. 419-423.

[129] J. Rodríguez-López and S. Romaguera, "The Hausdorff fuzzy metric on compact sets", *Fuzzy Sets and Systems,* 147(2) (2004), pp. 273-283.

[130] M. Imdad and J. Ali, "Some common fixed point theorems in fuzzy metric spaces", *Math. Commun.*, 11 (2006), pp. 153-163.

[131] W. Sintunavarat and P. Kumam, "Common fixed points for R-weakly commuting in fuzzy metric spaces", *Ann. Uni. Ferrara,* 58(2) (2012), pp. 389 - 406.

[132] I. Altun and D. Turkoglu, "Some fixed point theorems on fuzzy metric spaces with implicit relations", *Commun. Korean Math. Soc.*, 23 (1) (2008), pp. 111-124.

[133] S. Manro, S. S. Bhatia and S. Kumar, “Common fixed point theorems in fuzzy metric spaces”, *Ann. Fuzzy Math. Inform.*, 3(1) (2012), pp. 151-158.

[134] V. Pant and R. P. Pant, “Fixed points in fuzzy metric space for non compatible maps”, *Soochow J. Math.*, 33 (4) (2007), pp. 647-655.

[135] Y. Liu, J. Wu and Z. Li, “Common fixed points of single-valued and multivalued maps”, *Int. J. Math. Math. Sci.,* Vol. 2005, no. 19, pp. 3045-3055, 2005. doi: 10.1155/IJMMS.2005.3045

[136] M. Abbas, I. Altun and D. Gopal, “Common fixed point theorems for non compatible mappings in fuzzy metric spaces”, *Bull. Math. Anal. Appl.*, 1(2) (2009), pp. 47-56.

[137] S. Chauhan, W. Sintunavarat and P. Kumam, “Common Fixed Point Theorems for Weakly Compatible Mappings in Fuzzy Metric Spaces Using (JCLR) Property”, *Applied Mathematics*, 3(9) (2012), pp. 976-982.

[138] S. Chauhan, “Fixed points of weakly compatible mappings in fuzzy metric spaces satisfying common limit in the range property”, *Indian J. Math.*, 54(3) (2012), pp. 375-397.

[139] J. Ali, M. Imdad and D. Bahuguna, “Common fixed point theorems in Menger spaces with common property (E.A.)”, *Comput. Math. Appl.*, 60 (12) (2010), pp. 3152-3159.

[140] M. Jain and S. Kumar, “A common fixed point theorem in fuzzy metric space using the property (CLRg)”, *Thai J. Math.*, 14 (3) (2016), pp. 627-636.

[141] S. Chauhan, M. A. Khan and W. Sintunavarat, “Common fixed point theorems in fuzzy metric spaces satisfying ϕ-contractive condition with common limit range property”, *Abstr. Appl. Anal.*, Vol. 2013, Article ID 735217, 14 pages.

[142] M. Imdad, J. Ali and M. Hasan, “Common fixed point theorems in fuzzy metric spaces employing common property (E.A.)”, *Math. Comput. Modelling*, 55 (3-4) (2012), pp. 770-778.

[143] S. Chauhan, M. A. Khan and S. Kumar, “Unified fixed point theorems in fuzzy metric spaces via common limit range property”, *J. Inequal. Appl.*, 2013, **2013**:182.

[144] S. Sedghi, I. Altun and N. Shobe, “Coupled fixed point theorems for contractions in fuzzy metric spaces”, *Nonlinear Anal.*, 72(3) (2010), pp. 1298-1304.

[145] X. H. Zhu and J. Z. Xiao, “Note on “Coupled fixed point theorems for contractions in fuzzy metric spaces””, *Nonlinear Anal.*, 74 (2011), pp. 5475-5479.

[146] X. Q. Hu, “Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces”, *Fixed Point Theory Appl.*, Vol. 2011, Article ID 363716, 14 pages, 2011.

[147] X. Q. Hu, M. X. Zheng, B. Damjanović and X. F. Shao, “Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces”, *Fixed Point Theory Appl.*, 2013, **2013**: 220.

[148] S. Dalal and I. Masmali, “Coupled Fixed Point Theorems With CLRg Property in Fuzzy Metric Spaces”, *IJRRAS*, 15 (3) (2013), pp. 322-329.

[149] V. Berinde, “Generalized coupled fixed point theorems for mixed monotone mappings partially ordered metric spaces”, *Nonlinear Anal.*, 74 (2011), pp. 7347–7355.

[150] V. Berinde, “Coupled fixed point theorems for ϕ - contractive mixed monotone mappings in partially ordered metric spaces”, *Nonlinear Anal.*, 75(2012), pp. 3218 – 3228.

[151] J. Harjani, B. López and K. Sadarangani, “A fixed point theorem for Meir-Keeler contractions in ordered metric spaces,” *Fixed Point Theory Appl.*, 2011, **2011**:83.

[152] B. Samet, “Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces”, *Nonlinear Anal.*, 72 (2010), pp. 4508-4517.

[153] M. E. Gordji, Y. J. Cho, S. Ghods, M. Ghods, and M. H. Dehkordi, “Coupled fixed point theorems for contractions in partially ordered metric spaces and applications”, *Math. Prob. Eng.*, Vol. 2012, Article ID 150363, 20 pages, 2012.

[154] T. Abdeljawad, H. Aydi and E. Karapinar, “Coupled Fixed Points for Meir-Keeler Contractions in Ordered Partial Metric Spaces”, *Math. Prob. Eng.*, Vol. 2012, Article ID 327273, 20 pages, 2012.

[155] V. Berinde, and M. Pacurar, “Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces”, *Fixed Point Theory Appl.*, 2012, **2012**:115.

[156] B. Samet, C. Vetro and P. Vetro, “Fixed point theorems for α-ψ-contractive type mappings”, *Nonlinear Anal.*, 75 (2012), pp. 2154-2165.

[157] M. Mursaleen, S. A. Mohiuddine and R.P Agarwal, “Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces”, *Fixed Point Theory Appl.*, 2012, **2012**:228.

[158] E. Karapinar and R. Agarwal, “A note on ‘Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces’”, *Fixed Point Theory Appl.*, 2013, **2013**:216.

[159] M. Jain, K. Tas, S. Kumar and N. Gupta, “Coupled common fixed point results involving a $(\varphi,\ \psi)$-contractive condition for mixed g-monotone operators in partially ordered metric spaces”, *J. Inequal. Appl.*, 2013, **2013**:85.

[160] N. V. Luong and N. X. Thuan, “Coupled fixed point theorems in partially ordered G-metric spaces”, *Math. Comput. Modelling*, 55 (2012), pp. 1601-1609.

[161] H. K. Nashine, “Coupled common fixed point results in ordered G-metric spaces”, *J. Nonlinear Sci. Appl.*, 5(1) (2012), pp. 1-13.

[162] E. Karapinar, B. Kaymakçalan and K. Tas, “On coupled fixed point theorem on partially ordered G-metric spaces”, *J. Inequal. Appl.*, 2012, **2012**:200.

[163] S. A. Mohiuddine and A. Alotaibi, “On coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces”, *Abstr. Appl. Anal.*, Vol. 2012, Article ID 897198, 15 pages, 2012.

[164] M. Jain and K. Tas, “A unique coupled common fixed point theorem for symmetric (φ,ψ)-contractive mappings in ordered G-metric spaces with applications”, *J. Appl. Math.*, Vol. 2013, Article ID 134712, 13 pages, 2013.

[165] R. H. Haghi, Sh. Rezapour and N. Shahzad, “Some fixed point generalizations are not real generalizations”, *Nonlinear Anal.*, 74 (2011), pp. 1799-1803.

[166] W. Sintunavarat, Y. J. Cho and P. Kumam, “Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces”, *Fixed Point Theory Appl.*, 2011, **2011**: 81.

[167] N. Hussain, A. Latif and M. H. Shah, “Coupled and tripled coincidence point results without compatibility”, *Fixed Point Theory Appl.*, 2012, **2012**: 77.

[168] S. M. Alsulami, “Some coupled coincidence point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions”, *Fixed Point Theory Appl.*, 2013, **2013**:194.

[169] D. Turkoglu and M. Sangurlu, "Coupled fixed point theorems for mixed g-monotone mappings in partially ordered metric spaces", *Fixed Point Theory Appl.*, 2013, **2013**:348.

[170] N. Singh and R. Jain, "Coupled Fixed Point Results For Weakly Related Mappings In Partially Ordered Metric Spaces", *Bull. Iranian Math. Soc.*, 40 (1) (2014), pp. 29-40.

[171] F. Yan, Y. Su and Q. Feng, "A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations", *Fixed Point Theory Appl.*, 2012, **2012**:152.

[172] B. Samet, E. Karapinar, H. Aydi and V. Ć. Rajić, "Discussion on some coupled fixed point theorems", *Fixed Point Theory Appl.*, 2013, **2013**:50.

[173] D. Đorić, Z. Kadelburg and S. Radenović, "Coupled fixed point results for mappings without mixed monotone property", *Appl. Math. Lett.*, 25 (2012), pp. 1803-1808.

[174] S. Chandok and K. Tas, "An original coupled coincidence point result for a pair of mappings without MMP", *J. Inequal. Appl.*, 2014, **2014**:61.

BRIEF BIO-DATA OF THE RESEARCH SCHOLAR

Manish Jain is a research scholar in YMCA University of Science and Technology, Faridabad, India. He completed his B.Sc. (Non-Medical) from Ahir College, Rewari securing 1st position and M.Sc. (Mathematics) from M.D.U., P.G. Regional Centre, Rewari securing 2nd position. He also qualified CSIR-NET with All India Ranking 13.

With a vast teaching experience of nearly 12 years, he is presently working at Ahir College, Rewari as an Assistant Professor in Mathematics. His area of research is “Fixed Point Theory”. He has reviewed many research papers of various International Journals including some SCIE Journals and has contributed many research papers in his research field.

LIST OF PUBLICATIONS OUT OF THESIS

List of Published Papers

S. No.	Title of Paper	Name of Journal where published	No.	Volume & Issue	Year	Pages
1.	Coupled common fixed point results involving a (φ, ψ) -contractive condition for mixed g-monotone operators in partially ordered metric spaces, Article No. 285	Journal of Inequalities and Applications **(SCIE)**		Vol. 2012 & Issue 1	2012	19 pages
2.	Coupled Fixed Point Theorems for (φ, ψ)-Contractive Mixed Monotone Mappings in Partially Ordered Metric Spaces and Applications, Article ID 586096	International Journal of Analysis		Vol. 2014	2014	9 pages
3.	Coupled Fixed Point Theorems for Symmetric (ϕ, ψ) -weakly Contractive Mappings in Ordered Partial Metric Spaces	Journal of Mathematics and Computer Science		Vol. 7 & Issue 4	2013	276-292
4.	Coupled Fixed Point Theorems for Generalized Symmetric Contractions in Partially Ordered Metric Spaces and applications	Journal of Computational Analysis and Applications **(SCIE)**	3	Vol. 16	2014	438-454
5.	Coupled fixed point results for mappings involving (α, ψ) -weak contractions in ordered metric spaces and applications	Journal of Mathematics and Computer Science		Vol. 10 & Issue 1	2014	23-46

S. No.	Title of Paper	Name of Journal where published	No.	Volume & Issue	Year	Pages
6.	Coupled common fixed point results involving $(\phi, \psi) -$ contractions in ordered generalized metric spaces with application to integral equations, Article No. 372	Journal of Inequalities and Applications **(SCIE)**		Vol. 2013 & Issue 1	2013	22 pages
7.	A New Technique to Compute Coupled Coincidence Points, Article ID 652107	Chinese Journal of Mathematics		Vol. 2014	2014	6 pages
8.	A Note on "Common Coupled Fixed Point Results for Probabilistic φ - Contractions in Menger PGM-Spaces"	Communications in Nonlinear Analysis		Vol. 2 & Issue 1	2016	84-85
9.	Erratum: Coupled Fixed Point Results For Weakly Related Mappings In Partially Ordered Metric Spaces	Bulletin of the Iranian Mathematical Society **(SCIE)**	1	Vol. 42	2016	49-52
10.	A Discussion On Some Recent Coupled Fixed Point Results Via New Generalized Nonlinear Contractive Conditions	TWMS Journal of Applied and Engineering Mathematics **(ESCI - Web of Science)**	1	Vol. 7	2017	110-130

S. No.	Title of Paper	Name of Journal where published	No.	Volume & Issue	Year	Pages
11.	Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps along with CLRg Property in Fuzzy Metric Spaces, Article ID 961210	Journal of Applied Mathematics **(SCOPUS)**		Vol. 2012	2012	13 pages

List of Accepted Papers

S. No.	Title of Paper	Name of Journal where accepted	No.	Volume & Issue	Year of Acceptance
1.	Common Fixed Point Results for Various Mappings in Fuzzy Metric Spaces with Application (In Press)	Boletim da Sociedade Paranaense de Matemática **(ECSI – Web of Science & SCOPUS)**	...	...	2017
2.	Common fixed point results for w-compatible mappings along with (CLR_{ST}) property in fuzzy metric spaces	TWMS Journal of Applied and Engineering Mathematics **(ECSI – Web of Science & SCOPUS)**	...	...	2018

List of Conference Papers

S. No.	Title of Paper	Name of Conference	Date of Conference	Publication
1.	Coupled common fixed point results under new nonlinear contractions in ordered G-metric spaces	International Conference “ICSCMM-17” at KIET Group of Institutions, Ghaziabad, U.P.	Dec 22-23, 2017	Paper published in conference proceedings in: “Malaya Journal of Matematik”, Vol. S, No. 1, 2018, pages 5-13.

S. No.	Title of Paper	Name of Conference	Date of Conference	Publication
2.	Remarks on Some Recent Papers Concerning the Computation of Coupled Coincidence Points	National Conference "CPMSED-2015" held at JNU, New Delhi organized by Krishi Sanskriti	Nov 28, 2015	Paper published in the conference proceedings
3.	Improvement of some recent coupled coincidence point results under nonlinear contractions	National Conference "RSTTMI 2016" at YMCAUST, Faridabad	March 05-07, 2016	Paper published in the conference proceedings
4.	Generalization of a recent coupled coincidence point result for probabilistic φ-contraction	International Conference "RAMSA-2016" at JIIT, Noida	Dec 08-10, 2016	Abstract published in the conference souvenir
5.	Discussion on fixed point results via generalized weak $(\psi > \phi)$ –contractions in ordered metric spaces	International Conference "ICAA 2015" at AMU, Aligarh	Dec 19-21, 2015	Abstract published in the conference souvenir