(i) $A=\left\{\frac{1}{n}: n \in \mathrm{N}\right\}$.

(ii) **Z**, the set of integers.

(iii) **Q**, the set of rational numbers.

(iv) [0, 1]. (7)

6. (a) If f and g are continuous real valued function defined on a metric space (X, d), then prove that $f+g$ and af are also continuous functions, where a is any real number. (10)

(b) Define uniform continuous function on a metric space and prove that the function $f: [0, 1] \to \mathbf{R}$ such that $f(x)=x^2$ is uniformly continuous. (5)

7. (a) Discuss the convergence of the following improper integrals :

(i) $\int_0^\infty \sqrt{x} e^{-x} dx$.

(ii) $\int_0^1 \frac{x^n}{1-x} dx$. (10)

(b) Let (X, d) be a discrete metric space. Then prove that the only connected sets are empty set and singleton sets. (5)

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May-2026

B.Sc. (Hons.) Mathematics - VI Semester

Riemann Integral and Metric Space

(BMH24-605/MTU-310V)

Time : 3 Hours] [Maximum Marks : 75

Instructions :

1. *It is compulsory to answer all the questions (1.5 marks each) of Part-A in short.*
2. *Answer any four questions from Part-B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART–A

1. (a) If $f(x)=\frac{1}{x^2}$ on [1, 4] and P = {1, 2, 3, 4} be the partition of [1, 4], then compute $L(f, P)$. (1.5)

(b) State Mean Value theorem for integrals. (1.5)

(c) Give an example of a discontinuous function having only finite number of points of discontinuity on the interval [0, 1]. (1.5)

(d) If $(\mathbf{R}, d)$ be the usual metric space, find the derived set of set of natural numbers. (1.5)

(e) Test the convergence of the improper integral $\int_0^1 \frac{dx}{\sqrt{1-x}}$. (1.5)

(f) Give an example of a Cauchy sequence in a metric space. (1.5)

(g) Define complete metric space. (1.5)

(h) For which values of m and n, does the Beta function, defined as $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists. (1.5)

(i) If d be the usual metric space $d(x, y) = |x - y|$ for $x, y \in \mathbf{R}$ describe $S(0, 1)$, where $S(a, r)$ denotes the open sphere with radius r and centre a. (1.5)

(j) If (X, d) is a discrete metric space. Then prove that every subset of X is open. (1.5)

PART–B

2. (a) Show that the function f defined on [0, 1] by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}, \\ -1, & \text{if } x \in \mathbf{R}/\mathbf{Q} \end{cases}$$ is

bounded but not Riemann Integrable. (8)

(b) State and Prove first Fundamental theorem of calculus. (7)

3. (a) Prove the inequality $1 \leq \int_0^1 e^{x^2}dx \leq e$. (5)

(b) Test whether the following improper integrals are convergent or not :

(i) $\int_{-1}^1 \frac{dx}{x^2}$.

(ii) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. (10)

4. (a) Let $X = \mathbf{R}$ be the set of real numbers and $d : \mathbf{R} \times \mathbf{R}$ be a function defined as $d(x, y) = |x - y|$ for all $x, y \in \mathbf{R}$. Show that (X, d) is a metric space. (8)

(b) If A and B are subsets of a metric space (X, d) such that $A \subset B$, then prove the following :

(i) $A^\circ \subset B^\circ$, where C° denotes the interior of set C.

(ii) $\overline{A} \subset \overline{B}$, where $\overline{C}$ denotes the closure of set C. (7)

5. (a) Prove that a Cauchy sequence in a metric space is convergent iff it has a convergent subsequence. (8)

(b) Let $(\mathbf{R}, d)$ be the usual metric space with metric $d(x, y) = |x - y|$ for $x, y \in \mathbf{R}$. Determine the closure of the following subsets of $\mathbf{R}$: