

Determine whether f is continuous or not.

(ii) Show that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. (8)

6. (a) Define $f : [0, 1] \rightarrow \mathbb{R}$ by:

$$f(x) = \begin{cases} 1, & \text{for rational } x \text{ in } [0, 1] \\ -1, & \text{for irrational } x \text{ in } [0, 1]. \end{cases}$$

Prove that f is not integrable on $[0, 1]$. (7)

(b) Discuss the convergence of the following improper integrals :

(i) $\int_0^{\infty} \sqrt{x}e^{-x} dx.$

(ii) $\int_0^1 \frac{x^n}{1-x} dx.$ (8)

7. (a) Let (\mathbb{R}, d) be the usual metric space. Find whether or not the following subsets of \mathbb{R} are neighborhood of 1:

(i) $(0, 2).$ (ii) $[1, 2].$

(iii) $[0, 2] \setminus \left\{ \frac{1}{2} \right\}.$ (iv) $\mathbb{R}.$ (7)

(b) Let f and g be continuous real valued functions defined on a metric space (X, d) . Prove that $f + g$ is a continuous function. (8)

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B.SC. MATHEMATICS/B.SC. MATHEMATICS & COMPUTING VI SEMESTER

Riemann Integral & Metric Space (BMH-602)

Time : 3 Hours]

[Maximum Marks : 75

Instructions :

1. It is compulsory to answer all the questions (1.5 marks each) of Part-A in short.
2. Answer any **four** questions from Part-B in detail.
3. Different sub-parts of a question are to be attempted adjacent to each other.

PART-A

1. (a) Let f be a bounded function on a closed interval $[a, b]$. Define the upper Darboux integral $U(f)$ and the lower Darboux integral $L(f)$ of f over $[a, b]$. (1.5)

(b) Let f be a function defined on the interval $[0, m]$, where m is a positive integer by :

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer,} \\ 1, & \text{if } x \text{ is not an integer.} \end{cases}$$

Show that f is integrable on $[0, m]$. (1.5)

(c) Define an improper integral. Give an example. (1.5)

(d) Define the beta function and gamma function. (1.5)

(e) Give an example of a Riemann integrable function. (1.5)

(f) Define an open set. (1.5)

(g) Let (\mathbb{R}, d) be the usual metric space. Find the derived set of $[0, 1)$. (1.5)

(h) Let A and B be subsets of a metric space (X, d) . Suppose that $A \subseteq B$. Then what is the relation between A° and B° ? (Here, A° denotes the interior of the set A and B° denotes the interior of the set B .) (1.5)

(i) Define a homeomorphism. (1.5)

(j) Define a contraction mapping. (1.5)

PART-B

2. (a) Let $f(x) = \frac{1}{x^2}$ be a function defined on $[1, 4]$ and $P = \{1, 2, 3, 4\}$ be a partition of $[1, 4]$. Compute $U(f, P)$ and $L(f, P)$. (7)

(b) Prove that a bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that:

$$U(f, P) - L(f, P) < \epsilon.$$

(Here, $U(f, P)$ denotes the upper Darboux sum of f with respect to the partition P and $L(f, P)$ denotes the lower Darboux sum of f with respect to the partition P .) (8)

3. (a) Compute $\int_0^5 [x] dx$, where $[x]$ stands for greatest integer not greater than x . (7)

(b) Examine the convergence of the following integral :

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \quad (8)$$

4. (a) Let \mathbb{C} denotes the set of complex numbers. Define a function $d: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ by $d(z_1, z_2) = |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. Prove that (\mathbb{C}, d) is a metric space. (7)

(b) Let (\mathbb{R}, d) be the usual metric space with metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Determine the closure of the following subsets of \mathbb{R} :

(i) $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

(ii) The set of rational number \mathbb{Q} .

(iii) $[0, 1]$.

(iv) $(0, 1)$. (8)

5. (a) State and prove Banach fixed point theorem. (7)

(b) (i) Let (\mathbb{R}, d) be the usual metric space. Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ by:

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$