Sr. No. 753401

December 2023

M.Sc.(Maths) - IV SEMESTER

FUNCTIONAL ANALYSIS (MATH-21-851)

Time: 3 Hours

Max. Marks:75

Instructions:

1. *It is compulsory to answer all the questions (1.5 marks each) of Part -A in short.*
2. *Answer any four questions from Part -B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART -A

Q1 (a) Prove that in a normed linear space, every convergent sequence is a Cauchy sequence. (1.5)

(b) Show that every complete subspace M of a normed linear space X is closed. (1.5)

(c) If $T : X(K) \rightarrow Y(K)$ is a linear mapping in a normed linear space and if T is continuous at the origin, then it is continuous everywhere and continuity is uniform. (1.5)

(d) Prove that every compact subset of a normed linear space is complete. (1.5)

(e) Prove or disprove an inner product space is a normed linear space. (1.5)

(f) Let M be a non-empty subset of a Hilbert space H. Then, $M^{\perp}$ is a closed linear subspace of H. (1.5)

(g) Prove that the adjoint of a compact operator is compact. (1.5)

(h) Show that Weak limit of a sequence is unique. (1.5)

(i) Let H be a Hilbert space, an operator T on H is said to be normal if and only if $\| T^{*}x \| = \| Tx \|$. (1.5)

(j) Prove that the product of two self adjoint operators is self adjoint if and only if they commute. (1.5)

PART -B

Q2 (a) Let p be a real number such that $1 \leq p < \infty$ and let l_p denote the space of all sequences $x = \langle x_1, x_2, x_3, \ldots, x_n, \ldots \rangle$ of scalars such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Show that l_p is a Banach space under the norm (10)

$$\| x \|_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}.$$

(b) Let X be a normed linear space, Then, the mapping $f : X \rightarrow R$ such that $f(x) = \| x \|$ is continuous. (5)

Q3 (a) Let X and Y be normed linear spaces and let T be a linear transformation of X into Y. Show that the following statements are equivalent: (10)

(i) T is bounded;

(ii) T is uniformly continuous on X;

(iii) T is continuous at some point of X.

(b) Prove that every weakly convergent sequence in H is bounded. (5)

Q4 (a) Let M be a closed linear subspace of a normed linear space X and x_0 a vector not in M. Then, there exists a functional F in X^* such that (7)

$$F(M) = \{0\} \text{ and } F(x_0) \neq 0.$$

(b) Let X and Y be normed linear spaces and let D be a subspace of X. Then, a linear transformation $T : D \to Y$ is closed iff $x_n \in D$, $x_n \to x$ and $T(x_n) \to y \Rightarrow x \in D$ and $y = T(x)$. (8)

Q5 (a) Prove that a closed convex subset C of a Hilbert space H contains unique vector of smallest norm. (7)

(b) If M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$. (8)

Q6 (a) If T is an operator on a Hilbert space H, then T is normal if and only if its real and imaginary parts commute. (5)

(b) Let T be an operator on a Hilbert space H. Then, there exists a unique operator T^* on H such that $\langle Tx, \; y\rangle = \langle x, \; T^* y\rangle$, for all $x, \; y \in H$. The operator T^* is called the adjoint of the operator T. (10)

Q7 (a) In a finite dimensional normed linear space X, any subset M of X is compact if and only if M is closed and bounded. (8)

(b) Let $\{T_n\}$ be a sequence of compact linear operators from a normed space X into a Banach space Y. If $\{T_n\}$ is uniformly operator convergent, say $\| T_n - T \| \to 0$, then the limit operator T is compact. (7)

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