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323302

December, 2023

B.Sc. (MAC) / B.Sc. (H) MATHEMATICS-III SEMESTER

Group Theory (BMH -302A)

Time: 3 Hours **Max. Marks:75**

Instructions:

1. *It is compulsory to answer all the questions (1.5 marks each) of Part -A in short.*
2. *Answer any four questions from Part -B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART -A

Q1 (a) If G is a group, then inverse of every element of G is unique. (1.5)

(b) If $G = Z_4$. How many elements in Z_4 of order 4. (1.5)

(c) Show that in a group G, the equations $a.x = b$ and $y.a = b$ have unique solutions for all a, b in G. (1.5)

(d) If a is a generator of a cyclic group G, then a^{-1} is also a generator of G. (1.5)

(e) Let $H < G$ and $a, b \in G$. Show that $Ha \neq Hb$ *iff* $a^{-1}H \neq b^{-1}H$. (1.5)

(f) Show that there does not exist any non-abelian group G such that $o\left(G/Z(G)\right) = 37$, where $Z(G)$ is the centre of G. (1.5)

(g) Show that every homomorphic image of an abelian group is abelian. (1.5)

(h) Show that S_3 cannot be written as the internal direct product of its two non-trivial subgroups. (1.5)

(i) Show that any infinite cyclic group is isomorphic to $(Z, +)$. (1.5)

(j) If G is an abelian group, then the mapping $f : G \to G$ defined as $f(x) = x^5$, for all $x \in G$ is a homomorphism. (1.5)

PART -B

Q2 (a) Let G be a group and suppose there exist two relatively prime positive integers m and n such that $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$, for all $a, b \in G$. Show that G is abelian. (7)

(b) Let $f : G_1 \to G_2$ be a homomorphism. Let $a \in G_1$ be such that $o(a) = n$ and $o(f(a)) = m$. Show that $o(f(a)) / o(a)$ and f is one-one *iff* $m = n$. (8)

Q3 (a) Let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $N = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ and $G = \langle M, \ N \rangle$ be the group generated by matrices M and N under matrix multiplication, Then, find $o\left(\frac{G}{Z(G)}\right)$. (5)

(b) If $Z(G)$ is the centre of a group G such that $\frac{G}{Z(G)}$ is cyclic, then show that G is abelian. Is converse true? Justify your answer. (10)

Q4 (a) Find number of elements of order 7, 9 and 10 in C^*, where C^*-denotes set of all non-zero complex numbers. (8)

(b) If $G = Z_2 \times D_3$, find number of elements of all possible orders. (7)

Q5 (a) Let G be the group of non-zero complex numbers under multiplication and N the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplicative. (8)

(b) If G has elements of order d, then G has subgroup of order d. But converse need not be true. Justify your answer. (7)

Q6 (a) If G is a group and $G = Z_2 \times Z_4$, find all subgroups of $Z_2 \times Z_4$. (8)

(b) Write the class equation of Dihedral group D_3 and Quaternion group Q_4. (7)

Q7 If G is a finite group and $H_1, H_2, H_3, ..., H_n$ are normal subgroups of G such that $G = H_1.H_2.H_3....H_n$ and $o(G) = o(H_1).o(H_2).o(H_3)...o(H_n)$, prove that G is the direct product of $H_1, H_2, H_3, ..., H_n$. (15)
