Sr. No 323601

January– 2024

B.Sc. (H) Mathematics –VI SEMESTER

Riemann Integral and Metric Space (BMH–602)

Time: 3 Hours **Max. Marks:75**

Instructions:

1. *It is compulsory to answer all the questions (1.5 marks each) of Part –A in short.*
2. *Answer any four questions from Part –B in detail.*
3. *Different sub–parts of a question are to be attempted adjacent to each other.*
4. *Symbols used in this paper have their usual meaning.*

PART –A

Q1 (a) Prove that if f is real valued function defined on the closed and bounded interval $[a,b]$ and P be any partition of $[a,b]$, then $L(f,P) \le U(f,P)$. (1.5)

(b) State Mean Value theorem for integrals. (1.5)

(c) Give an example of a discontinuous function having only finite number of points of discontinuity on the interval $[0,5]$. (1.5)

(d) Let A be a subset of a metric space (X,d), then define the interior of A. (1.5)

(e) Test the convergence of the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$. (1.5)

(f) Let (R,d) be the discrete metric space. Then find the derived set of the set of irrational numbers. (1.5)

(g) Define cauchy sequence in a metric space. (1.5)

(h) Prove that every infinte times differentiable function is Riemann integrable on the finite interval $[a,b]$. (1.5)

(i) Define contraction mapping. (1.5)

(j) Let (X,d) be an infinite discrete metric space. Then prove that every function f from (X,d) to any metric space (Y,d^*) is continuous. (1.5)

PART –B

Q2 (a) Prove that if f and g are real valued bounded functions defined on $[a,b]$and P be any partition of $[a,b]$, then $U(f+g,P) \le U(f,P) + U(g,P)$, where $U(h,P)$ denotes the upper Darboux sum of the function h for the partition P. (7)

(b) Evaluate $\int_a^b e^x\, dx$ by using the limit of Riemann sums. (8)

Q3 (15)

Test whether the following improper integrals are convergent or not

(i) $\int_{-\infty}^{0} \frac{1}{p^2+q^2x^2} dx$

(ii) $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx,\qquad m > 0, n > 0.$

Q4 (a) Let X be the set of all real valued functions defined on $[a, b]$ and let d be a function from $X \times X \to \mathbb{R}$ such that $d(f, g) = \sup |f(x) - g(x)|$, where $x \in [a, b]$, for all $f, g \in X$, then prove that (X, d) is a metric space. (7)

(b) Show that in a metric space (X, d), the complement of every singleton set is open. (8)

Q5 (a) Prove that a point a is a limit point of a subset A of a metric space (X, d) if and only if there exists a sequence $< a_n >$ of points of A distinct from a such that $< a_n >$ converges to a. (7)

(b) Let (R, d) be a usual metric space. Prove or disprove that the following subsets of R are neighbourhood of 2. (8)

(i) $(0,2)$
(ii) $[1,2]$
(iii) $[0,2] - 1$
(iv) R

Q6 (a) Show that every cauchy sequence is bounded in a metric space. (7)

(b) Evaluate the following integrals: (8)

(i) $\int_0^{\pi/2} \cos^{20}\theta \, \sin^{24}\theta \, d\theta$

(ii) $\int_0^{\infty} \frac{dx}{4+x^2}$.

Q7 (a) Prove that the function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ for all $x \in \mathbb{R}$ is not uniformly continuous on $\mathbb{R}$. (7)

(b) Prove the inequality $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2} dx \leq \frac{2}{\pi}$. (8)

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