6. (a) Discuss the maxima and minima of

$f(x, y) = x^3y^2 (1 - x - y)$. (8)

(b) Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point (1, –2, –1). In the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$. (7)

7. (a) For what values of k, the equations (7)

$$x + y + z = 1,$$

$$2x + y + 4z = k,$$

and $4x + y + 10z = k^2$

have (i) Unique solution (ii) Infinite number of solutions (iii) No solution and solve them completely.

(b) If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, then find A^n in terms of A and I. (8)

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013101

December 2023

B.Tech. (Mechanical Engineering) 1st SEMESTER

Mathematics-I (Calculus and Linear Algebra) (BSC-103A)

Time : 3 Hours] [Max. Marks : 75

Instructions :

1. *It is compulsory to answer all the questions (1.5 marks each) of Part-A in short.*
2. *Answer any four questions from Part-B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART–A

1. (a) Test the convergence of the improper integral $\int_{-1}^{1} \frac{dx}{x^2}$. (1.5)

(b) What is relation between Beta and Gamma functions? (1.5)

(c) State Mean value theorems. (1.5)

(d) Explain "ALTERNATING SERIES" with example. (1.5)

(e) Evaluate (1.5)

$$\lim_{x\to\infty} \frac{x^n}{e^x}.$$

(f) Test the convergence of the following infinite seque[illegible]e : (1.5)

$$\{\sqrt{n+1}-\sqrt{n}\}.$$

(g) Find grad ϕ for the function $\phi(x, y, z) = 3xy + y^3z$ at the point (1, –2, 1). (1.5)

(h) State Cayley-Hamilton Theorem. (1.5)

(i) Find the Eigenvalues of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. Hence, find the matrix whose Eigenvalues are $\frac{1}{6}$ and –1. (1.5)

(j) Prove that a square matrix A and its transpose A^T have the same eigenvalues. (1.5)

PART–B

2. (a) Find the evolute of the rectangular hyperbola $xy = c^2$. (8)

(b) The portion of the curve $y = \frac{x^2}{2}$ cut off by the straight line $y = \frac{3}{2}$ is revolved about the y–axis. Find the surface area of revolution. (7)

3. (a) Evaluate $\int_0^{\pi/2} \sin^8\theta \cos^2\theta \, d\theta$. (5)

(b) Using Taylor's theorem, prove that

$$x - \frac{x^3}{6} < \sin x - \frac{x^3}{6} + \frac{x^5}{120}, \text{ for } x > 0.$$ (10)

4. (a) Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$. (5)

(b) Discuss the nature of the series (10)

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^2 x^3 + \ldots\infty, \text{ for } x > 0.$$

5. (a) If $a_0, a_1, a_2, \ldots, a_n$ are real numbers such that (5)

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0,$$

then there exists at least one x in (0, 1) such that

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0.$$

(b) Find the Fourier series for (10)

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.