7. (a) State and Prove Banach fixed point theorem. (10)

(b) Prove the inequality $1 \leq \int_0^1 e^{x^2}\, dx \leq e$. (5)

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May, 2023

B.Sc. (H) Mathematics -VI Semester

Riemann Integral and Metric Space (BMH-602)

Time : 3 Hours] [Max. Marks : 75

Instructions :

1. *It is compulsory to answer all the questions (1.5 marks each) of Part-A in short.*
2. *Answer any four questions from Part-B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART–A

1. (a) Prove that if f is integrable on the interval $[a, b]$ and $f(x) \leq 0$ for all $x \in [a, b]$, then $\int_a^b f\, dx \leq 0$. (1.5)

(b) State Mean Value theorem for integrals. (1.5)

(c) Give an example of a discontinuous function having only finite number of points of discontinuity on the interval [0,1]. (1.5)

(d) Let A and B be any two subsets of a metric space (X, d), then define the distance between A and B. (1.5)

(e) Test the convergence of the improper *integral* $\int_1^\infty \frac{dx}{x^2 - 3x + 2}$. (1.5)

(f) Let (R, d) be the discrete metric space. Then find the derived set of the set of natural numbers. (1.5)

(g) Define complete metric space. (1.5)

(h) Prove that every differentiable function is Riemann integrable on the finite interval $[a, b]$. (1.5)

(i) Give an example of a contraction mapping. (1.5)

(j) Let (X, d) be an infinite discrete metric space. Then prove that every function f from (X, d) to any metric space (Y, d^*) is continuous. (1.5)

PART–B

2. (a) Prove that if f and g are real valued bounded functions defined on $[a, b]$ and P be any partition of $[a, b]$, then $L(f + g, P\} \leq L(f, p) + L(g, p)$, where $L(h, P)$ denotes the lower Darboux sum of the function h for the partition P. (7)

(b) Evaluate $\int_1^4 x\, dx$ by using the limit of Riemann sums. (8)

3. Test whether the following improper integrals are convergent or not (15)

(a) $\int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}} dx$.

(b) $\int_0^1 \frac{dx}{x^{5/2}(1+x^2)} dx$.

4. (a) Let (X, d) be a metric space and k be a positive real number. Then prove that (X, d_k) is a metric space where $d_k\{x, y) = kd(x, y)$ for all $x, y \in X$. (7)

(b) Prove that every contraction mapping $f : (X, d) \to (X, d)$ is uniformly continuous. (8)

5. (a) Show that if (X, d) is a metric space and A is the finite subset of X, then the closure of A is the set itself. (7)

(b) Let (R, d) be a usual metric space. Prove or disprove that the following subsets of R are neighbourhood of 1.

(i) (0, 2)

(ii) [1, 2]

(iii) [0, 2] –1

(iv) R. (8)

6. (a) Show that every trivial (discrete) metric space is a complete metric space. (7)

(b) Evaluate the following integrals : (8)

(i) $\int_0^{\pi/2} \cos^{10}\theta \sin^{14}\theta \, d\theta$.

(ii) $\int_0^{\infty} \frac{dx}{1+x^2}$.