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May, 2023

B.Sc. (H) Mathematics VIth SEMESTER

Number Theory (DEMH-604)

Time : 3 Hours] [Max. Marks : 75

Instructions :

1. *It is compulsory to answer all the questions (1.5 marks each) of Part-A in short.*
2. *Answer any four questions from Part-B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART–A

1. (a) Find the *lcm* and *gcd* of 272 and 1479. (1.5)

(b) Find the remainder if 3^{40} is divided by 23. (1.5)

(c) Find all positive integral solutions of the Diophantine equation $5x + 3y = 52$. (1.5)

(d) Justify with the help of an example that converse of Fermat's theorem is not true. (1.5)

(e) Check whether the set {11, –3, –4, 7, 18, 22} forms a CRS (mod 6) or not? (1.5)

(f) Evaluate $\mu(270)$. (1.5)

(g) Find the highest power of 7 contained in 1000! (1.5)

(h) Define primitive root. Also, give an example. (1.5)

(i) State Fermat's last theorem. (1.5)

(j) Using the linear cipher $C \equiv P + 11 \ (mod\ 26)$, encrypt the message NUMBER THEORY. (1.5)

PART–B

2. (a) Solve the congruence $259x \equiv 5(mod\ 11)$. (5)

   (b) State and prove fundamental theorem of arithmetic. (10)

3. (a) Prove that a linear congruence $ax \equiv b(mod\ m)$, where a is not a multiple of m, has a solution if and only if $gcd(a,m)$ divides b. (5)

   (b) Let x and y be two real numbers and m be an integer, then prove the following properties : (10)

   I. $[x-y] \le [x]-[y] \le [x-y]+1$.

   II. $\left[\frac{[x]}{m}\right]=\left[\frac{x}{m}\right]$.

4. State and prove Chinese remainder theorem. Use this theorem to find all the integers that give the remainders 2, 6, 5 when divided by 5, 7, 11 respectively. (15)

5. (a) State and prove F. Merten's Lemma. (5)

   (b) Define Euler's function. Prove that if p is a prime and $k > 0$, then (10)

$$\phi(p^k) = p^k\left(1-\frac{1}{p}\right).$$

6. (a) Find all the quadratic residues and non-residues of 7. (5)

   (b) If p is an odd prime and a, b be any integers coprime to p, then prove that the following properties of Legendre symbol hold (10)

   I. $\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

   II. $\left(-\frac{1}{p}\right)=(-1)^{\frac{p-1}{2}}$.

   III. $\left(\frac{ab^2}{p}\right)=\left(\frac{a}{p}\right)$.

   IV. $\left(\frac{1}{p}\right)=1$.

7. (a) Encipher the message HAPPY DAYS ARE HERE using the autokey cipher with seed Q. (5)

   (b) Prove that the positive primitive solutions of $x^2 + y^2 = z^2$, with y even are $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$, where r and s are integers of opposite parity with $r > s > 0$ and $(r, s) = 1$. (10)
