Roll No. ....

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323402

May 2023

B.Sc. (H) Mathematics - IV SEMESTER

Ring Theory and Linear Algebra (BMH-402A)

Time: 3 Hours **Max. Marks:75**

Instructions:

1. *It is compulsory to answer all the questions (1.5 marks each) of Part -A in short.*
2. *Answer any four questions from Part -B in detail.*
3. *Different sub-parts of a question are to be attempted adjacent to each other.*

PART -A

Q1 (a) Let Z be the ring of integers and let 6Z be the ideal, then show that $Z/_{6Z}$ is not an integral domain. (1.5)

(b) Find the units of the ring Z_n. (1.5)

(c) Show that, the characteristic of an integral domain is either zero or prime number. (1.5)

(d) State second Isomorphism theorem for rings. (1.5)

(e) Let $\emptyset$ be a ring homomorphism from a ring R to a ring S. Then show that, $Ker\ \emptyset$ is an ideal of R. (1.5)

(f) Show that, the set (1.5)

$$S = \{A \in M_2(R) \ : \ |A| = 0\}$$

under the usual operations is not a vector space.

(g) Show that the union of two subspaces of a vector space V need not form a subspace of V. (1.5)

(h) Differentiate between Linearly independent and linearly dependent vectors. (1.5)

(i) Define the following terms: (1.5)

a. Linear Transformation

b. Linear Operator

c. Dialation Operator

(j) Let , $T : R^3 \to R^2$ be the linear transformation defined by (1.5)

$$T(x,y,z) = (x, y - 3z)$$

Find $N(T)$ and $R(T)$.

PART -B

Q2 (a) Give an example of the following: (5)

1. A non-commutative ring without multiplicative identity.
2. A finite non-commutative ring.
3. An infinite integral domain with non-zero characteristic.
4. A ring with unity which has a subring without unity.
5. A right ideal which is not a left ideal.

(b) Let $Z[x]$ be the ring of all polynomials with integer coefficients. Let I be a subset of $Z[x]$ consisting of all polynomials with constant term zero. Then show that, I is an ideal of $Z[x]$. Further show that, (5)

$$I = \langle x \rangle$$

(c) Prove the followings: (5)

1. Both $2Z$ and $3Z$ are maximal ideals in Z.
2. In a Boolean ring R every prime ideal I is a maximal ideal.

Q3 (a) Determine all ring homomorphism from Z_{12} *to* Z_{30}. (10)

(b) Let R be a commutative ring and suppose that, $Char\ R = p$, where p is a prime number. Then show that, the mapping, (5)

$$f : R \rightarrow R,$$

defined by

$$f(x) = x^p \quad ; \ \forall\ x \in R$$

is a ring homomorphism.

Q4 (a) Show that R^n forms a vector space over the field of real numbers. (7)

(b) Show that, (8)

1. $\{1,\ \sqrt{3},\ \sqrt{5}\}$ is linearly independent in R over the field of rational numbers.
2. $\{1,\ \sqrt{3},\ \sqrt{5},\ \sqrt{15}\}$ is linearly independent in R over the field of rational numbers.

Q5 (a) State and prove "Sylvester's Theorem". (8)

(b) Suppose that, V and U are two vector spaces over the field F such that, $\{x_1, x_2, x_3, ---, x_n\}$ is a basis for V. Then show that for $\{z_1, z_2, z_3, ---z_n\} \in U$ there exists exactly one linear transformation (7)

$$T : V \rightarrow U$$

such that,

$$T(x_i) = z_i \ ; \ \forall\ 1 \le i \le n$$

Q6 (a) Define Principal Ideal Domain and hence show that the ring of integers $(Z, +, \cdot)$ is a principal ideal domain. (8)

(b) Show that, (7)

$$\frac{Z[x]}{\langle x \rangle} \cong Z$$

Q7 (a) State and prove Replacement Theorem. (7)

(b) Let (4)

$$T : V \rightarrow U$$

be an invertible linear transformation. Then show that V is a finite dimensional vector space if and only if U is a finite dimensional vector space.

(c) Find the matrix representation of the linear operator, (4)

$$T : R^3 \rightarrow R^3$$

given by

$$T(x, y, z) = (2y + z, x - 4y, 3x)$$

relative to the basis

$$\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$$
